ENDEXTENSIONS IN BOUNDED ARITHMETIC AND COMPUTATIONAL COMPLEXITY

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Abstract. Let N be an elementary extension of $\mathbb{N}$ and $n \in N - \mathbb{N}$. We prove that $PTC(n)$ has no proper endextension of Δ_1^b-LLIND and consider conditions that a model of bounded arithmetic has a proper end extension.

Let N be an elementary extension of the set of natural numbers $\mathbb{N}$ and M be a substructure of N which is a model of PA^-, then M has a proper endextension of PA^-. If M satisfies PA, then it is well known that M also has a proper endextension of PA. In this paper, we consider conditions that a model M of a theory T in bounded arithmetic has a proper endextension of some subtheory of T.

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1. Polynomial time closure

Let n be a nonstandard element i.e. $n \in N - \mathbb{N}$. $PTC(n)$ denotes the polynomial time closure of $\{n\}$ in N, then $PTC(n)$ is a model of T^0_2. If $\mathbf{P} = \mathbf{NP}$, then $PTC(n)$ is a model of S_2. In this section, we prove

Theorem 1. *$PTC(n)$ has no proper endextension which satisfies $\Delta^b_1 - LLIND$.*

Let $T(e, x; \alpha)$ be an oracle Turing machine satisfying the condition that for any oracle Turing machine $T'(x; \alpha)$ there are infinitely many natural numbers e such that

$$T(e, x; \alpha) = T'(x; \alpha).$$

Let $T(e, x; \alpha)(t)$ denote the content of the output tape of the machine T at the time t of the computation with an input x. Then $y = T(e, x; \alpha)(|z|)$ can be written as a $\Delta^b_1(\alpha)$-formula which we write by

$$y = \{e\}(x, |z|; \alpha).$$

The polynomial time closure $PTC(n)$ of $\{n\}$ is the set

$$\{\{e\}(n, |n|^{|e|}; \emptyset)|\ e \in \mathbb{N}\}.$$

Assume that $PTC(n)$ has a proper endextension L of $\Delta^b_1 - LLIND$. Let $m \in L - PTC(n)$. Since L is an endextension of $PTC(n)$, for all $e \in \mathbb{N}$ we have $m > 2^{|n|^e} \in$

$PTC(n)$. Now we consider the following infinite set

$$\alpha = \{x \in PTC(n) \mid x < |n|\}.$$

Since L is an endextension of $PTC(n)$, α is defined in L by a Δ_1^b-formula with the parameters n and m, i.e.

$$\begin{aligned}\alpha =& \{x \in L \mid L \models x < |n| \wedge x = \{e\}(n, |n|^{|e|}; \emptyset) \text{ for some } e \in \mathbb{N}\} \\ =& \{x \in L \mid L \models x < |n| \wedge \exists e < ||m||(x = \{e\}(n, |n|^{|e|}; \emptyset) \wedge \forall y < e(x \neq \{y\}(n, |n|^{|y|}; \emptyset)))\}.\end{aligned}$$

Then $\mathbb{N}$ is defined in L by a Δ_1^b-formula with parameters n and m,

$$\mathbb{N} = \{e \in L \mid L \models e < ||m|| \wedge \exists x < |n|(x = \{e\}(n, |n|^{|e|}; \emptyset) \wedge \forall y < e(x \neq \{y\}(n, |n|^{|y|}; \emptyset)))\}$$

Let $\Phi(e, m, n)$ be the defining Δ_1^b-formula of $\mathbb{N}$ in L, then $LLIND(\Phi(x, m, n))$ does not hold in L.

2. Endextensions

In this section, we try to construct an endextension L of $PTC(n)$ which contains 2^n. Since $\exists x(x = 2^n)$ holds in L but not in $PTC(n)$, L cannot be an elementary extension of $PTC(n)$. So many tools in model theory, omitting type arguments, internal ultrapower etc. which give endextensions are of no use. Let $M = \{x \in N \mid x < y$ for some $y \in PTC(n)\}$. Then M is closed under the smash function, hence M is a model of S_2.

First we define L to be the set of all bounded subsets of $PTC(n)$ defined by Σ^b-formulas, in other words, $\alpha \in L$ if and only if there exists a Σ^b-formula $\Phi(x, y)$ and $a \in PTC(n)$ such that

$$\alpha = \{x \in PTC(n) \mid PTC(n) \models \Phi(x, a) \wedge x < a\}.$$

For such $\alpha, \Phi(x, y)$ and $a \in PTC(n)$, let

$$\alpha_M = \{x \in M \mid M \models \Phi(x, a) \wedge x < a\}.$$

If $\mathbf{P} = \mathbf{NP}$, then $PTC(n)$ is a Σ^b-elementary substructure of M, hence $\alpha = \alpha_M \cap PTC(n)$. Since α_M is definable in N, there exists a $c_\alpha \in N$ such that

$$\alpha_M = \{x \in N \mid bit(c_\alpha, x) = 1\}.$$

By identifying $\alpha \in L$ with c_α, we can consider $L \subset N$. Let $\alpha = \{n\} \in L$, then $c_\alpha = 2^n$, so L contains 2^n.

Lemma 1. *If* $\mathbf{P} = \mathbf{NP}$, *then* L *is an endextension of* $PTC(n)$.

Proof. Let $c_\alpha \in L$. Then there is a Σ^b-formula $\Phi(x, y)$ and a $a \in PTC(n) \subset M$ such that $\alpha = \{x \in PTC(n) \mid PTC(n) \models \Phi(x, a) \wedge x < a\}$. Let $b \in PTC(n)$ be such that $c_\alpha < b$, then we have

$$\alpha = \{x \in PTC(n) \mid PTC(n) \models \Phi(x, a) \wedge x < |b|\}.$$

Since we are assuming $\mathbf{P} = \mathbf{NP}$, $PTC(n)$ is a model of S_2, hence $c_\alpha \in PTC(n)$.

Lemma 2. *If* $\mathbf{P} = \mathbf{NP}$, *then* $\alpha \in L$ *implies* $\lfloor \frac{1}{2}\alpha \rfloor \in L$.

Proof. Let $\Phi(x, y)$ and $a \in PTC(n)$ define α, i.e.

$$\alpha = \{x \in PTC(n) \mid PTC(n) \models \Phi(x, a) \wedge x < a\},$$

then

$$\lfloor \frac{1}{2}\alpha \rfloor = \{x \in PTC(n) \mid PTC(n) \models \Phi(x + 1, a) \wedge x < a - 1\}.$$

Lemma 3. *If* $\mathbf{P} = \mathbf{NP}$, *then* $\alpha, \beta \in L$ *implies* $\alpha \sharp \beta \in L$.

Proof. Let $\Phi(x, y)$ and $a \in PTC(n)$ (resp. $\Psi(x, y)$ and $b \in PTC(n)$) define α (resp. β), then

$$\alpha\sharp\beta = \{x \in PTC(n) \mid PTC(n) \models x = max(\alpha) + max(\beta)\}$$

$$= \{x \in PTC(n) \mid PTC(n) \models \exists y < a \exists z < b(x = y + z \wedge \Phi(y) \wedge \Psi(z)$$

$$\wedge \forall v < a(y < v \rightarrow \neg\Phi(v)) \wedge \forall w < b(z < w \rightarrow \neg\Psi(w)) \wedge x < a + b\}.$$

Lemma 4. *If* $\mathbf{P} = \mathbf{NP}$*, then* $\alpha, \beta \in L$ *implies* $\alpha + \beta \in L$*.*

Proof. Let $\Phi(x, y), \Psi(x, y)$ and $a, b \in PTC(n)$ be as in the proof of Lemma 3. Then $\alpha + \beta$ is defined by the following bounded formula and $max(a, b) + 1$.

$$((\Phi(x, y) \triangle \Psi(x, y)) \wedge \exists i < x(\neg\Phi(i, y) \wedge \neg\Psi(i, y) \wedge \forall j < x(i < j \rightarrow (\Phi(j, y) \triangle \Psi(j, y)))))$$

$$\vee(\neg(\Phi(x, y) \triangle \Psi(x, y)) \wedge \exists i < x(\Phi(i, y) \wedge \Psi(i, y) \wedge \forall j < x(i < j \rightarrow (\Phi(j, y) \triangle \Psi(j, y)))))$$

where $\Phi \triangle \Psi$ denotes $(\Phi \wedge \neg\Psi) \vee (\neg\Phi \wedge \Psi)$.

Next we consider multiplication on L. To prove that L is closed under multiplication, we need more assumption than $\mathbf{P} = \mathbf{NP}$.

Lemma 5. *Assume that* $\mathbf{P} = \mathbf{PSPACE}$*, then* $\alpha, \beta \in L$ *implies* $\alpha \cdot \beta \in L$*.*

Proof. $\alpha \cdot \beta$ is computable by a **PSPACE** machine with oracles α and β, more precisely, there exists an oracle **PSPACE**-machine $S(x; \alpha, \beta)$ such that

$$x \in \alpha \cdot \beta \text{ if and only if } S(x; \alpha, \beta).$$

Since $\alpha, \beta \in L$ and $\mathbf{P} = \mathbf{PSPACE}$, $S(x, \alpha, \beta)$ is poly-time computable, hence $\alpha \cdot \beta \in L$.

Lemma 6. *Assume that* $\mathbf{P} = \mathbf{PSPACE}$*, then* L *is a model of* $\Sigma_0^b - LIND$*.*

Proof. Let $\Phi(x)$ be a Σ_0^b-formura, then there is a $\Sigma_0^{1,b}$ formula $\phi(x)$ such that

$$L \models \Phi(x) \text{ if and only if } PTC(n) \models \phi(x).$$

Assume that $PTC(n) \models \phi(0) \wedge \neg\phi(t)$ for some $t \in PTC(n)$. Since we are assuming $\mathbf{P} = \mathbf{PSPACE}$, we can find $x \in PTC(n)$ by a **PSPACE**-machine such that $PTC(n) \models \phi(x) \wedge \neg\phi(x+1)$, as contended.

Conjecture. *Assume that* $\mathbf{P} = \mathbf{PSPACE}$*, then* L *is a model of* $\Delta_1^b - LLIND$*.*

If we prove the Conjecture, then by Theorem 1, we can conclude $\mathbf{P} \neq \mathbf{PSPACE}$.

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