Distributivity of $({}^{<\kappa}\kappa)^L$ and $0^\sharp$

Hiroshi Sakai

May 27, 2002

Abstract

In this paper, we shall examine distributivity of $({}^{<\kappa}\kappa)^L$. The motivation of our examination is the following theorem of Foreman-Magidor-Shelah [4]: If $0^\sharp$ exists then for every nontrivial partial order $\mathbb{P} \in L$, forcing with $\mathbb{P}$ over V adds a real.

On the other hand, if κ is regular in L then it is not easy to find forcing over L which preserves regularity of κ and forces that $({}^{<\kappa}\kappa)^L$ is not κ-distributive. In this paper, among other things, we shall give an sufficient condition for $N \models$ " $({}^{<\kappa}\kappa)^L$ is not λ-distributive " for a transitive N of ZFC and then we see that, if κ is a weakly compact cardinal of order 1 in L, then there is a forcing over L which forces $\kappa = \omega_2$ and $({}^{<\kappa}\kappa)^L$ is not σ-distributive.

1 Introduction

Partial orders play a central part in forcing and so we often think about properties of partial orders such as chain condition, closedness, distributivity, properness.... In general, these properties are not absolute. For example, if $M \subseteq N$ are transitive models of ZFC s.t. $({}^{\omega}2)^M \subsetneq ({}^{\omega}2)^N$, then the partial order $({}^{<\omega_1}\omega_1)^M$ is σ-closed in M but not in N. Moreover if $({}^{\omega}2)^M \subsetneq ({}^{\omega}2)^N$ then, except for trivial partial orders, every partial order in M is not σ-distributive in N. In this paper, we examine nonabsoluteness of distributivity of partial orders, in particular, of ${}^{<\kappa}\kappa$. For each ordinal κ, ${}^{<\kappa}\kappa$ is the partial order s.t.

$${}^{<\kappa}\kappa := \{f \mid f \text{ is a function} \wedge dom(f) < \kappa \wedge ran(f) \subseteq \kappa\},$$

and for each f and g in ${}^{<\kappa}\kappa$,

$$f \leq g \text{ iff } f \text{ is an end extension of } g.$$

If κ is regular, it can be easily seen that ${}^{<\kappa}\kappa$ is κ-closed.

Distributivity of partial orders is defined as follows.

Definition 1.1
Assume κ is an ordinal and $\mathbb{P}$ is a partial order. Then

$$\mathbb{P} \text{ is } \kappa\text{-distributive.} \overset{\text{def}}{\Longleftrightarrow} \text{If } \langle D_\xi \mid \xi < \lambda\rangle,\ \lambda < \kappa, \text{ is a sequence of dense open subsets of } \mathbb{P} \text{ then } \bigcap_\xi D_\xi \text{ is dense open in } \mathbb{P}.$$

We call ω_1-distributive partial orders σ-distributive.

So every κ-closed partial order is κ-distributive. That $\mathbb{P}$ is κ-distributive is equivalent to that forcing with $\mathbb{P}$ adds no sequence of ordinals of length less than κ. In particular, if $\mathbb{P}$ is σ-distributive then $\mathbb{P}$ adds no real.

As closedness, distributivity is not absolute. For example, assume $M \subseteq N$ are transitive models of ZFC, κ is regular in M but not in N, and $\mathbb{P} \in M$ is a partial order s.t. $M \models$ " $\mathbb{P}$

is κ-distributive but not κ^+-distributive ". Then $\mathbb{P}$ is not κ-distributive in N. But if κ is regular also in N then the problem is more complicated. Related to nonabsoluteness of distributivity, the following striking theorem was shown by Foreman, Magidor and Shelah. Our examination of nonabsoluteness of distributivity is motivated by this theorem. For the definition and basic properties of $0^\sharp$, see Section 2.

Theorem 1.2 (Foreman-Magidor-Shelah [4])
Assume $0^\sharp$ exists. Then for every nontrivial partial order $\mathbb{P} \in L$, forcing with $\mathbb{P}$ over V adds a real.

So if $0^\sharp$ exists then every constructible partial order is not σ-distributive in V. There, the reverse implication of the theorem is conjectured.

Conjecture
Assume, for every nontrivial partial order $\mathbb{P} \in L$, forcing with $\mathbb{P}$ over V adds a real. Then $0^\sharp$ exists.

If $0^\sharp$ exists then V satisfies large cardinal axioms restricted to L. For example, if $0^\sharp$ exists then there is a definable elementary embedding from L to L. On the other hand, if $0^\sharp$ does not exist then V has no such strong properties for L. Moreover Covering Theorem implies that if $0^\sharp$ does not exist then V is subjected to restrictions about cofinalities. For example, if α is singular cardinal in V then α is singular also in L and $\alpha^+ = (\alpha^+)^L$. So whether $0^\sharp$ exists or not can be seen as whether V is transcendental relative to L or not. So the above conjecture states that " every constructible partial order adds a real " means the transcendence of V relative to L.

By the way, even for single L-regular cardinal κ, it is not easy to find a forcing over L which preserves regularity of κ and forces that $({}^{<\kappa}\kappa)^L$ is not κ-distributive. In Stanley [8], it is shown that if κ is weakly compact in L then there is a forcing $\mathbb{S}$ over L s.t. $\Vdash_{\mathbb{S}}$ " κ is regular " and $\Vdash_{\mathbb{S}}$ " $\mathbb{P}$ is not κ-distributive " for many $\mathbb{P} \in L$ including $({}^{<\kappa}\kappa)^L$. But, for example, if κ is a successor cardinal of a singular cardinal in L, then, as far as we know, there is no forcing over L which preserves regularity of κ and forces that $({}^{<\kappa}\kappa)^L$ is not κ-distributive. This means that we do not know any forcing over L which forces " $\exists\kappa(\kappa$ is a successor cardinal of a singular cardinal $\wedge$ $({}^{<\kappa}\kappa)^L$ is not κ-distributive) ". It is an open problem, for example, whether " $\kappa = \omega_{\omega+1}$ and $({}^{<\kappa}\kappa)^L$ is not κ-distributive " implies that " $0^\sharp$ exists " or not.

On the other direction, $\mathbb{S}$ constructed in Stanley [8] has the following property: $\Vdash_{\mathbb{S}}$ " $({}^{<\kappa}\kappa)^L$ is not κ-distributive ", but for every $\lambda < \kappa$, $\Vdash_{\mathbb{S}}$ " $({}^{<\kappa}\kappa)^L$ is λ-distributive ". In general, if $\mathbb{P} \in L$ is κ-closed in L and $\lambda < \kappa$, then it is not easy to find a forcing over L which preserves regularity of κ and forces that $\mathbb{P}$ is not λ-distributive.

In this paper, we ristrict ourselves to the partial order ${}^{<\kappa}\kappa$. First, in Theorem 3.3, we shall give a sufficient condition for $N \models ({}^{<\kappa}\kappa)^M$ is not λ-distributive ", where $\lambda \leq \kappa$ and $M \subseteq N$ are transitive models of ZFC. And then, we shall give examples satisfying this condition. Among these examples, in Theorem 4.13, we see

" Assume κ is a weakly compact cardinal of order 1 in L. Then there is a forcing over L which forces $\kappa = \omega_2$ and $({}^{<\kappa}\kappa)^L$ is not σ-distributive. "

So if " ZFC + (there is a weakly compact cardinal of order 1 in L) " is consistent then " ZFC + ($0^\sharp$ does not exist) + ($\kappa = \omega_2 \wedge ({}^{<\kappa}\kappa)^L$ is not σ-distributive) " is consistent, and so " ZFC + ($\kappa = \omega_2 \wedge ({}^{<\kappa}\kappa)^L$ is not σ-distributive) " does not implies " $0^\sharp$ exists ".

But we do not know any forcing $\mathbb{T}$ over L s.t. for some κ, $\Vdash_{\mathbb{T}}$ " $\kappa = \omega_3$ and $({}^{<\kappa}\kappa)^L$ is not ω_2-distributive ". In this direction, it is open whether " $\kappa = \omega_3$ and $({}^{<\kappa}\kappa)^L$ is not ω_2-distributive " implies " $0^\sharp$ exists ".

As mentioned above, it is included in the results of Stanley [8] that if κ is weakly compact in L, then there is a forcing over L which preserves regularity of κ and forces that $({}^{<\kappa}\kappa)^L$ is not κ-distributive. But the forcing constructed there is not aimed to add C of Theorem 3.3. So, in Example 1 of Section 4, we shall construct a forcing in view of adding C of Theorem 3.3. C of Theorem 3.3 can be also added using the forcing developed in Gitik-Magidor-Woodin [7]. But here we use a reverse Easton forcing to add C of Theorem 3.3. The forcing constructed here gives another proof of Theorem 1 of Gitik-Magidor-Woodin [7].

Acknowledgement: The author would like to thank Professor Katsuya Eda for giving precious advice. In particular, the author studied many things in his seminar.

2 $0^\sharp$ and Silver indiscernibles

In this section, we see basic properties of Silver indiscernibles. For details and backgrounds, see, for example, Kanamori [6]. Before defining Silver Indiscernibles, we shall define definable Skolem functions, Skolem terms and Skolem hulls. Let $\mathcal{L}_\in$ be the language of Set Theory. Let $\mathfrak{M} = \langle M, E \rangle$ be a structure of $\mathcal{L}_\in$ s.t. there is a well order $<$ of M which is definable over $\mathfrak{M}$. (Note that if α is a limit ordinal then $\langle L_\alpha, \in \rangle$ satisfies this property.)

First we shall define definable Skolem functions. For each $(n+1)$-ary formula φ of $\mathcal{L}_\in$, define $h_\varphi^{\mathfrak{M},<} : {}^nM \to M$ as follows.

If $\mathfrak{M} \models$ " $\exists v \varphi[v, x_1, ..., x_n]$ " then

$h_\varphi^{\mathfrak{M},<}(x_1, ..., x_n) :=$ the $<$-least y s.t. $\mathfrak{M} \models$ " $\varphi[y, x_1, ..., x_n]$ "

otherwise $h_\varphi^{\mathfrak{M},<}(x_1, ..., x_n) := 0$.

We call $h_\varphi^{\mathfrak{M},<}$ the definable Skolem function for φ in $\mathfrak{M}$. If $<$ is in some sense canonical then we call $h_\varphi^{\mathfrak{M},<}$ the canonical Skolem function for φ in $\mathfrak{M}$.

Next we shall define Skolem terms. For each $(n+1)$-ary formula $\varphi(v_0, v_1, ..., v_n)$ of $\mathcal{L}_\in$, let t_φ be an n-ary function symbol. Then let $\mathcal{L}_{\in,t} := \mathcal{L}_\in \cup \{t_\varphi \mid \varphi$ is a formula of $\mathcal{L}_\in\}$. Then, by interpreting t_φ as $h_\varphi^{\mathfrak{M},<}$, $\langle M, E, h_\varphi^{\mathfrak{M},<}\rangle_\varphi$ becomes a structure of the language $\mathcal{L}_{\in,t}$. We call a term of $\mathcal{L}_{\in,t}$ "Skolem term" or simply "term". For each term $t(v_1, ..., v_n)$ let $t^{\mathfrak{M},<}[x_1, ..., x_n]$ denote $t^{\langle M,E,h_\varphi^{\mathfrak{M},<}\rangle_\varphi}[x_1, ..., x_n]$.

Finally, we shall define the Skolem hull. For each $X \subseteq M$, let

$Skull^{\mathfrak{M},<}(X) := \{t^{\mathfrak{M},<}[\vec{x}] \mid t$ is a term and $\vec{x} \in [X]^{<\omega}\}$.

Note that if $<'$ is another well order of M which is definable over $\mathfrak{M}$, then $Skull^{\mathfrak{M},<}(X) = Skull^{\mathfrak{M},<'}(X)$. So we omit $<$ and simplify $Skull^{\mathfrak{M}}(X)$. We call $Skull^{\mathfrak{M}}(X)$ the Skolem hull of X in $\mathfrak{M}$. It can be easily seen

$\langle Skull^{\mathfrak{M}}(X), E \restriction Skull^{\mathfrak{M}}(X)\rangle \prec \mathfrak{M}$.

In this paper, we sometimes identify structures with their universes. Let $<_L$ be the canonical well order of L. As mentioned above, if α is a limit ordinal then $<_L \cap L_\alpha$ is definable over L_α. We call $<_L \cap L_\alpha$ the canonical well ordering of L_α and let $h_\varphi^{L_\alpha}$ and t^{L_α} denote $h_\varphi^{L_\alpha, <_L \cap L_\alpha}$ and $t^{L_\alpha, <_L \cap L_\alpha}$ respectively.

Next we shall define indiscernible sequences.

Definition 2.1
Let M be a set structure of some language and $\langle S, < \rangle$ be a linear order s.t. $S \subseteq M$. Then we say $\langle S, < \rangle$ is an indiscernible sequence for M iff the following holds.

Assume $n \in \omega$ and $\varphi(v_1, ..., v_n)$ is a formula where $v_1, ..., v_n$ are all of its free variables. Then

$$M \models \text{``}\ \varphi[\alpha_1, ..., \alpha_n]\ \text{''} \Longleftrightarrow M \models \text{``}\ \varphi[\beta_1, ..., \beta_n]\ \text{''}$$

whenever $\alpha_1 < \alpha_2 < ... < \alpha_n$ and $\beta_1 < \beta_2 < ... < \beta_n$ are sequences of elements of S.

If the order of sequence S is $\in$, we sometimes omit $\in$ and call S an indiscernible sequence for M.

Intuitionally, the class of Silver indiscernibles I is the closed unbounded class of ordinals s.t. I is an indiscernible sequence for L. In general, the satisfaction relation of a proper class cannot be defined and " I is an indiscernible sequence for L " cannot be formulated. But if there is a proper class E of ordinals s.t. for each $\alpha < \beta$ in E, $L_\alpha \prec L_\beta$ then this problem of formulation can be solved. If there is such E then $\langle L_\alpha, \sigma_{\alpha,\beta} \mid \alpha < \beta \text{ in } E \rangle$ is an elementary chain with its direct limit L, where $\sigma_{\alpha,\beta}$ is an inclusion map from L_α to L_β. So if we define, for each formula $\varphi(v_1, ..., v_n)$ and $x_1, ..., x_n$ in L,

$$L \models \text{``}\ \varphi[x_1, ..., x_n]\ \text{''} \overset{\text{def}}{\Longleftrightarrow} \text{for some } \alpha \in E,\ \ x_1, ..., x_n \text{ are in } L_\alpha, \text{ and } L_\alpha \models \text{``}\ \varphi[x_1, ..., x_n]\ \text{''}$$

then by the reflection principle, for any given formula $\varphi(v_1, ..., v_n)$ and given $x_1, ...x_n$ in L,

$$\varphi^L(x_1, ..., x_n) \quad \text{iff} \quad L \models \text{``}\ \varphi[x_1, ..., x_n]\ \text{''}.$$

(φ^L is the relativisation of φ to L.) Moreover " I is an indiscerbile sequence for L " can be formulated using this definition. If such class E exists then we say that the satisfaction of L is definable. Note that if $\alpha \in E$ then α is a limit ordinal. So, for each term $t(v_1, ..., v_n)$ and $x_1, ..., x_n$, define $t^L[x_1, ..., x_n]$ similarly and, for each given class $X \subseteq L$, let $Skull^L(X)$ be $\{t^L[x_1, ..., x_n] \mid x_1, ..., x_n \in X\}$. Note that $Skull^L(X) = \bigcup_{\alpha \in E} Skull^{L_\alpha}(X \cap L_\alpha)$.

Note that the definition of $L \models$ " $\varphi\,[\,x_1, ..., x_n\,]$ " is not depend on the choice of E. Let E' be another class of ordinals s.t. for each $\alpha < \beta$ in E', $L_\alpha \prec L_\beta$. Let $\bar{E}$ and $\bar{E}'$ be the closures of E and E' respectively. Then for each $\eta \in \bar{E}$, $\langle L_\alpha, \sigma_{\alpha,\beta} \mid \alpha < \beta < \eta$, α and β in $E\rangle$ is an elementary chain with the direct limit L_η. So for each $\alpha < \beta$ in $\bar{E}$, $L_\alpha \prec L_\beta$. Similar is true for E' and $\bar{E}'$. Let $F := \bar{E} \cap \bar{E}'$. Note that F is a proper class s.t. for each $\alpha < \beta$ in F, $L_\alpha \prec L_\beta$. Then it is easily seen that the definition of $L \models$ " $\varphi\,[\,x_1, ..., x_n\,]$" using F is same as that using E. This is also true for F and E'. So the definition using E is same as that of E'.

Now we shall define the class of Silver indiscernibles. Assume the satisfaction of L can be defined. Then a class I of ordinals are called the class of Silver indiscernibles iff,

1. I is an indiscernible sequence for L.
2. $Skull^L(I) = L$.
3. I is closed unbounded.

Later we see that the class of Silver indiscernibles is unique if exists. First we examine basic properties of the class of Silver indiscernibles.

Notation:
If $\vec{\alpha}$ and $\vec{\beta}$ are finite sequences of ordinals then $\vec{\alpha} < \vec{\beta}$ means that $max(\vec{\alpha}) < min(\vec{\beta})$. Similarly if $\vec{\alpha}$ is a finite sequence of ordinals and $\beta \in On$ then let $\vec{\alpha} < \beta$ and $\beta < \vec{\alpha}$ mean $max(\vec{\alpha}) < \beta$ and $\beta < min(\vec{\alpha})$ respectively.

Lemma 2.2

Assume the satisfaction of L is definable and I is the class of Silver indiscernibles. Then

1. *Assume t is a term, $\vec{j}$, $\vec{k}$ are finite increasing sequences in I s.t. $\vec{j} < \vec{k}$ and $t^L[\vec{j},\vec{k}] \in L_{min(\vec{k})}$. Let i be the least element of I s.t. $\vec{j} < i$. Then $t^L[\vec{j},\vec{k}] \in L_i$.*
2. *Assume t is a term, $\vec{j}$, $\vec{k}$ are finite increasing sequences in I s.t. $\vec{j} < \vec{k}$ and $t^L[\vec{j},\vec{k}] \in L_{min(\vec{k})}$. Then for every finite increasing sequence $\vec{l}$ in I s.t. $\vec{j} < \vec{l}$ and $\vec{l}$ has the same length as $\vec{k}$, $t^L[\vec{j},\vec{k}] = t^L[\vec{j},\vec{l}]$.*
3. *If $i < j$ are in I then $L_i \prec L_j$.*
4. *If $i \in Lim(I)$ then $Skull^L(I \cap i) = Skull^{L_i}(I \cap i) = L_i$.*
5. *If κ is a cardinal then $\kappa \in Lim(I)$.*

[proof]

1. We can assume $min(\vec{k}) > i$. Let $\vec{l}$ be a finite increasing sequence in I s.t. $\vec{j} < \vec{l}$ and $min(\vec{l})$ is in $Lim(I)$. Then, by the indiscernibility, $t^L[\vec{j},\vec{l}] \in L_{min(\vec{l})}$. Because $min(\vec{l}) \in Lim(I)$, there is $i' < min(\vec{l})$ s.t. $i' \in I$ and $t^L[\vec{j},\vec{l}] \in L_{i'}$. Then, by the indiscernibility, $t^L[\vec{j},\vec{k}] \in L_i$. $\square$

2. First we shall show 2 under the assumption that $\vec{k} < \vec{l}$. Assume $t^L[\vec{j},\vec{k}] \neq t^L[\vec{j},\vec{l}]$. Let $\kappa > min(\vec{k})$ be a cardinal and $\langle \vec{k}_\xi \mid \xi \in \kappa \rangle$ be a sequence of finite sequences in I s.t. $\vec{k}_0 = \vec{k}$ and $\vec{k}_\xi < \vec{k}_\eta$ for every $\xi < \eta$. Then, by the indiscernibility, $t^L[\vec{j},\vec{k}_\xi] \neq t^L[\vec{j},\vec{k}_\eta]$ for every $\xi < \eta$. But by 1, $t^L[\vec{j},\vec{k}_\xi] \in L_i$ for every $\xi \in \kappa$, where i is the successor element of $max(\vec{j})$ in I. This contradicts to $|L_i| < \kappa$.

If $\vec{k} \not< \vec{l}$, let $\vec{i}$ be a finite sequence in I s.t. $\vec{k} < \vec{i}$, $\vec{l} < \vec{i}$ and $\vec{i}$ has the same length as $\vec{k}$ and $\vec{l}$. Then $t^L[\vec{j},\vec{k}] = t^L[\vec{j},\vec{i}] = t^L[\vec{j},\vec{l}]$. $\square$

3. Let $x \in L_i$ and φ be a formula. Assume $x = t^L[\vec{k},\vec{l}]$ where $\vec{k}$ and $\vec{j}$ are finite sequences in I s.t. $\vec{k} < i \leq \vec{l}$. By 2, we can assume $j < \vec{l}$. Then by indiscernibility, $L_i \models$ " $\varphi[x]$ " iff $L_j \models$ " $\varphi[x]$ ". $\square$

4. By 3, I is a proper class s.t. for each $i < j$ in I, $L_i \prec L_j$. So as mentioned before, $t^L[x] = t^{L_i}[x]$ for each $x \in L_i$. This implies $Skull^L(I \cap i) = Skull^{L_i}(I \cap i)$. $Skull^L(I \cap i) = L_i$ is clear by 2. $\square$

5. Assume $\kappa \notin Lim(I)$. Let i be the least element of $Lim(I)$ s.t. $i > \kappa$. Then clearly $|I \cap i| < \kappa$. By 4, $L_\kappa \subseteq Skull^L(I \cap i)$. This contradicts to $|L_\kappa| = \kappa$. $\square$

Next we shall show the uniqueness of the class of Silver indiscernibles.

Proposition 2.3

Assume the satisfaction of L is definable and, I and J are classes of ordinals which satisfy the properties of the class of Silver indiscernibles. Then $I = J$.

[proof]

Let i be the least element in $I \backslash J \cup J \backslash I$. Without loss of genericity, we can assume $i \in I \backslash J$. Then $i = t^L[\vec{k},\vec{l}]$ for some term t and finite sequences $\vec{k}$, $\vec{l}$ in J s.t. $\vec{k} < i < \vec{l}$. By the leastness of i, $\vec{k}$ is a sequence in I, and by 2 of Lemma 2.2, we can assume that $min(\vec{l})$ is large enough and $\vec{l}$ is a sequence in $I \cap J$. (Note that I and J are closed unbounded.) So, by the indiscernibility of I, $i' = t^L[\vec{k},\vec{l}]$ for each $i' \in I$ s.t. $\vec{k} < i' < \vec{l}$. This is a contradiction. $\square$

Next we shall think about the statement " there is the class of Silver indiscernibles ". In general, the existence of certain class can not be formulated in a first order sentence. But in this case, there is a satisfactory solution.

First we see more about the class of Silver indiscernibles. Assume the satisfaction of L is definable and I is the class of Silver indiscernibles. For each uncountable regular cardinal κ, let $I_\kappa := I \cap \kappa$. Then by Lemma 2.2, I_κ satisfies the following.

1. I_κ is closed unbounded in κ.
2. $Skull^{L_\kappa}(I_\kappa) = L_\kappa$.
3. I_κ is an indiscernible sequence for L_κ.

Conversely assume for each regular κ, there is I_κ satisfying the above 1-3. Then as in the proof of Lemma 2.2 and Prop. 2.3, we can easily show the following.

- For each regular κ, I_κ is the unique set which satisfies the above 1-3.
- If $\lambda < \kappa$ are regular cardinals then $\lambda \in Lim(I_\kappa)$ and $I_\lambda = I_\kappa \cap \lambda$.
- If $\lambda < \kappa$ are regular cardinals then $L_\lambda \prec L_\kappa$.

So the satisfaction of L is definable and, if we let $I := \bigcup\{I_\kappa \mid \kappa \text{ is regular}\}$, then I becomes the class of Silver indiscernibles.

So let " there is the class of Silver indiscernibles " denote the first order sentence stating that for each regular cardinal κ, there is I_κ satisfying the above 1-3.

Finally we shall define $0^\sharp$. Assume there is the class of Silver indiscernibles I. Let $\langle i_\xi \mid \xi < On\rangle$ be the increasing enumeration of I. Let $\mathcal{L}'$ be the language obtained by adding to $\mathcal{L}_\in$ ω constant symbols $\{c_n \mid n \in \omega\}$. Then, if we interpret each c_n as i_n, $\langle L_{i_\omega}, \in, i_n\rangle_n$ becomes the structure of $\mathcal{L}'$. Then, by assigning each sentence of $\mathcal{L}'$ to a natural number in the canonical way, the theory of $\langle L_{i_\omega}, \in, i_n\rangle_n$ can be seen as a real. Let $0^\sharp$ be this real. Note that, by the indiscernibility of I, $0^\sharp$ codes the theory of $\langle L, \in, i_\xi\rangle_{\xi \in On}$. Using the completeness theorem, it can be easily seen I is definable over $L[0^\sharp]$. Let " $0^\sharp$ exists " mean that there is the class of Silver indiscernibles, i.e. for each regular κ there is I_κ satisfying the above 1-3. In this paper, if $0^\sharp$ exists, then let I be the class of Silver indiscernibles.

If $0^\sharp$ exists then cardinalities and cofinalities in V are very different from those of L. Assume $i \in I$. Then by, 5 of Lemma 2.2 and the indiscernibility of I, i is a regular limit cardinal in L. So every $i \in I$ is inaccessible in L. Moreover it can be easily seen i is a weakly compact cardinal in L.

Indeed, if $0^s harp$ exists then V satisfies large cardinal axioms restricted to L. For example, let $f : I \to I$ be an order preserving embedding and $\pi_f : L \to L$ be the function s.t. $\pi_f(t^L[\vec{i}\,]) = t^L[\,f(\vec{i})\,]$ for each $\vec{i} \in {}^{<\omega}I$ and term t. Then clearly π_f is an elementary embedding from L to L. So if $0^\sharp$ exists then there is a definable elementary embedding from L to L. On the other hand, if $0^\sharp$ does not exist, then not only V has not such strong properties for L but also Covering Theorem implies that, as we see below, V is subjected to restrictions about cofinalities.

Theorem 2.4 (Devlin-Jensen [3])
Assume $0^\sharp$ does not exist. Then for every uncountable set of ordinals X, there is $Y \in L$ s.t. $X \subseteq Y$ and $|X| = |Y|$.

This theorem implies, for example, the following. Assume $0^\sharp$ does not exist, then

- If α is singular then α is also singular in L.
- If α is singular then $\alpha^+ = (\alpha^+)^L$.

For the first statement, let $X \subset \alpha$ be the cofinal subset of α s.t. $|X| < \alpha$. Then there is $Y \in L$ s.t. $X \subseteq Y$ and $|Y| = |X| \cdot \omega_1 < \alpha$. Then $o.t.(Y) < \alpha$ and Y is cofinal in α. So α is singular in L. For the second statement, assume $(\alpha^+)^L$ is not a cardinal. Then $cf((\alpha^+)^L) < \alpha$. So there is a cofinal $X \subseteq (\alpha^+)^L$ s.t. $|X| < \alpha$. By Covering Theorem, there is $Y \in L$ s.t. $X \subseteq Y$ and $|Y| = |X| \cdot \omega_1$. In particular, Y is cofinal in $(\alpha^+)^L$ and $o.t.(Y) < \alpha$. This contradicts to that $(\alpha^+)^L$ is regular in L.

3 A sufficient condition for $N \models$ " $({}^{<\kappa}\kappa)^L$ is not κ-distributive ".

Let $M \subseteq N$ be a transitive models of ZFC, κ be a regular cardinal in M, and λ be a regular cardinal in N. In this section, we shall give one sufficient condition for $N \models$ " $({}^{<\kappa}\kappa)^M$ is not λ^+-distributive " and then, using this, we shall show that if $0^\sharp$ exists then $({}^{<\kappa}\kappa)^L$ is not σ-distributive for each L-regular cardinal κ. Note that for each partial order $\mathbb{P}$, the least α s.t. $\mathbb{P}$ is not α-distributive is the successor cardinal of a regular cardinal.

To give a sufficient condition for $N \models$ " $({}^{<\kappa}\kappa)^M$ is not λ^+-distributive ", we need some preparations.

Lemma 3.1
Assume $M \subseteq N$ are transitive models of ZFC, κ is regular in M and $\lambda < \kappa$ is regular in N. Then the following are equivalent.
1. *$({}^{<\kappa}\kappa)^M$ is not λ^+-distributive in N.*
2. *In N, there is a dense subset T of $({}^{<\kappa}\kappa)^M$ s.t. if $\langle t_\xi \mid \xi < \lambda\rangle$ is a decreasing sequence of elements of T then $\bigcup_{\xi<\lambda} t_\xi \notin ({}^{<\kappa}\kappa)^M$.*

[proof] We shall work in N. If $|\kappa|^N = \lambda$ then clearly both 1 and 2 hold. So assume $|\kappa|^N > \lambda$. Let $\langle s_\alpha \mid \alpha < \beta\rangle$ be an enumeration of $({}^{<\kappa}\kappa)^M$ in N, where $\beta \geq \kappa$ is a cardinal in N.
$(1 \Rightarrow 2)$. Because $({}^{<\kappa}\kappa)^M$ is not λ^+-distributive, for some $\delta \leq \lambda$ there is a sequence of dense open subsets of $({}^{<\kappa}\kappa)^M$, $\langle A_\xi \mid \xi < \delta\rangle$ s.t.

- $A_\xi \supseteq A_\eta$ whenever $\xi < \eta < \delta$.
- $\bigcap_{\xi<\delta} A_\xi = \emptyset$.

Let $T \subseteq ({}^{<\kappa}\kappa)^M$ be s.t. for each $t \in ({}^{<\kappa}\kappa)^M$,
$t \in T \iff$ for some $\xi < \delta$,
$t \in A_\xi$ and, for each proper initial segment s of t, $s \notin A_\xi$.
It can be easily seen that T satisfies the property of 2.
$(2 \Rightarrow 1)$. Let T be a dense subset of $({}^{<\kappa}\kappa)^M$ which satisfies the property of 2. Let $F : \lambda \times \beta \to T$ be the function defined as follows. By induction as for the lexicographical order $\lhd$ on $\lambda \times \beta$, we shall define $F(\xi, \alpha)$. Assume $F(\xi', \alpha')$ was defined for each $(\xi', \alpha') \lhd (\xi, \alpha)$. Then let $F(\xi, \alpha)$ be $t \in T$ s.t. $t \leq s_\alpha$ in $({}^{<\kappa}\kappa)^M$ and $t \neq F(\xi', \alpha')$ for each $(\xi', \alpha') \lhd (\xi, \alpha)$. Because $\{(\xi', \alpha') \mid (\xi', \alpha') \lhd (\xi, \alpha)\}$ has cardinality $max(\lambda, |\alpha|)$ and there are β-many $t \in T$ s.t. $t \leq s_\alpha$, so $F(\xi, \alpha)$ can be defined. Then F is an injection and $\{F(\xi, \alpha) \mid \alpha < \beta\}$ is dense for each $\xi < \lambda$. Let $A_\xi := \{t \in ({}^{<\kappa}\kappa)^M \mid \exists \alpha < \beta (t \leq F(\xi, \alpha))\}$. Then A_ξ is dense open for each $\xi < \lambda$ and $\bigcap_{\xi<\lambda} A_\xi = \emptyset$. $\square$

Definition 3.2
Let λ be a limit ordinal and C_1, C_2 be closed unbounded in λ. We say C_1 is faster than C_2 iff for some $\eta < \lambda$, $C_1 \backslash \eta \subseteq C_2$.

Note: In this paper, if $cf(\lambda) = \omega$ and C is a cofinal subset of λ of order type ω then we say C is closed unbounded.

Now we shall give a sufficient condition for $N \models$ " $({}^{<\kappa}\kappa)^M$ is not λ-distributive ".

Theorem 3.3
Assume $M \subseteq N$ are transitive models of ZFC and $\lambda < \kappa$ are ordinals s.t. λ is regular in N and κ is regular in M. If there is $C \in N$ s.t.

1. *C is closed unbounded in κ.*
2. *If $\eta \in Lim(C)$ and $cf(\eta)^N = \lambda$ then η is regular in M.*
3. *If $\eta \in Lim(C)$ and $cf(\eta)^N = \lambda$ then $\eta \cap C$ is faster than every closed unbounded subset of η in M.*

Then $({}^{<\kappa}\kappa)^M$ is not λ^+-distributive in N.

[proof]
We shall work in N. If $cf(\kappa) \leq \lambda$ then clearly $({}^{<\kappa}\kappa)^M$ is not λ^+-distributive. So assume $cf(\kappa) > \lambda$. Let C be a closed unbounded subset of κ satisfying 1-3. For each $\eta \in C$, let η_+ denote the successor element of η in C. Then let

$T := \{t \in ({}^{<\kappa}\kappa)^M \mid \exists \eta \in C(dom(t) = \eta + 1 \ \wedge \ t(\eta) = \eta_+)\}$.

We shall show T satisfies the property of 2 of Lemma 3.1. Clearly T is dense in $({}^{<\kappa}\kappa)^M$. Let $\langle t_\xi \mid \xi < \lambda \rangle$ be a decreasing sequence in T. We show $t := \bigcup_{\xi<\lambda} t_\xi$ is not in M. For each $\xi < \lambda$, let $\eta_\xi := dom(t_\xi) - 1$ and $\eta := dom(t) = sup\{\eta_\xi \mid \xi < \lambda\}$.

Assume $t \in M$. Let $\bar{t}$ be the function s.t. $\bar{t}(\alpha) = t(\alpha) \ mod \ \eta$. Then $\bar{t} \in M$. Because $\eta \in Lim(C)$ and $cf(\eta) = \lambda$, η is regular in M. So $B := \{\beta < \eta \mid \bar{t}\,``\beta \subseteq \beta\}$ is closed unbounded in η and $B \in M$. So $C \cap \eta$ is faster than B. Then there is $\xi < \lambda$ s.t. $(\eta_\xi)_+ \in B$. But $\bar{t}(\eta_\xi) = t(\eta_\xi) = t_\xi(\eta_\xi) = (\eta_\xi)_+$. This is a contradiction. □

Corollary 3.4
Assume $0^\sharp$ exists. Then for each L-regular κ, $({}^{<\kappa}\kappa)^L$ is not σ-distributive.

[proof]
The proof is divided into two cases. Let I be the class of Silver indiscernibles and $\langle i_\xi \mid \xi < On \rangle$ be its increasing enumeration. For each term t, if t is interpreted in L, then we omit L in t^L and simply write t.

Case 1 $\kappa = i_\xi$ for some ξ of uncountable cofinality.

We shall show $I \cap \kappa$ satisfies the properties of Theorem 3.3 with $\lambda = \omega$ and $M = L$. 1 and 2 was shown in Section 2. Assume $i < i_\xi$, $i \in Lim(I)$, and $B \in L$ is closed unbounded in i. Then there are a term t and finite sequences in I, $\vec{j}$ and $\vec{k}$, s.t. $\vec{j} < i < \vec{k}$ and $t[\vec{j}, i, \vec{k}] = B$. We claim that if $l \in I$ and $\vec{j} < l < i$ then $l \in B$. First note that $B \cap l = t[\vec{j}, l, \vec{k}]$: Assume $\alpha < l$ and $\alpha = t_\alpha[\vec{j}_\alpha, \vec{k}_\alpha]$ where t_α is a term and $\vec{j}_\alpha \leq \alpha < \vec{k}_\alpha$ are finite sequences in I. We can also assume $\vec{k} < \vec{k}_\alpha$. Then, by the indiscernibility, $t_\alpha[\vec{j}_\alpha, \vec{k}_\alpha] \in t[\vec{j}, l, \vec{k}]$ iff $t_\alpha[\vec{j}_\alpha, \vec{k}_\alpha] \in t[\vec{j}, i, \vec{k}]$. So $B \cap l = t[\vec{j}, l, \vec{k}]$. Then by the indiscernibility, $B \cap l$ is unbounded in l. So $l \in B$.

Case 2 Otherwise.

We shall show κ has cofinality ω. So we can assume $\kappa \notin Lim(I)$. Let i_ξ be the largest element of I s.t. $i_\xi < \kappa$. Then for each $n \in \omega$, let

$X_n := Skull^L(i_\xi \cup \{i_\xi, i_{\xi+1}, i_{\xi+2}, ..., i_{\xi+n}\})$, and
$\alpha_n := Sup(\kappa \cap X_n)$.

Note that $X_n \in L$ and $|X_n|^L = i_\xi$. Because κ is regular in L, so $\alpha_n < \kappa$ for each $n \in \omega$. Recall that if $i \in Lim(I)$ then $Skull^L(i \cap I) = L_i$. So $\bigcup_{n\in\omega} X_n = L_{i_{\xi+\omega}}$. So $sup\{\alpha_n \mid n \in \omega\} = \kappa$ and $cf(\kappa) = \omega$. □

We shall end this section with the following observation of Theorem 3.3. Assume $M = L$ and $N \models$ " $0^\sharp$ does not exist ". Then Covering Theorem implies that if $\kappa > \omega_2$ is regular in N and there is $C \subseteq \kappa$ satisfying the properties 1-3 of Theorem 3.3 then $\kappa = \lambda^+$. So, using Theorem 3.3, we can not find a transitive model N of ZFC s.t. $N \models$ " $0^\sharp$ does not exist " and $N \models$ " $\kappa = \omega_3 \ \wedge \ ({}^{<\kappa}\kappa)^L$ is not ω_2-distributive ". In this sense, Theorem 4.13 is the best possible. Here we think about T constructed in the proof of Theorem 3.3.

Notation:

1. Assume f is a function s.t. $ran(f) \subseteq On$ and $\alpha \in On$. Then let $f_{\,mod\ \alpha}$ be the function s.t. $f_{\,mod\ \alpha}(x) = f(x)\ mod\ \alpha$.
2. Assume $\alpha \in On$ and f and g are functions from α to α. Then we say f dominates g iff $\exists \eta < \alpha \forall \xi > \eta\ (\ f(\xi) \geq g(\xi)\ )$.

Let κ, λ, M and N be as in Theorem 3.3. Then T constructed in the proof has the following property.

- Assume $\langle t_\xi \mid \xi < \lambda \rangle$ is a decreasing sequence in T. Let $t := \bigcup_{\xi<\lambda} t_\xi$ and $\alpha := dom(t)$. Then $t_{\,mod\ \alpha}$ is not dominated by each $f \in {}^\alpha\alpha \cap M$.

This property is stronger than the property of 2 of Lemma 3.1. We see that if $M = L$, $\lambda = \omega$ and κ is sufficiently large regular cardinal, then the existence of T satisfying above property implies $N \models$ " $0^\sharp$ exists ".

Proposition 3.5
Assume $0^\sharp$ does not exists. Assume $\kappa > \omega_2$ is regular and T is dense in $({}^{<\kappa}\kappa)^L$. Then there is a decreasing sequence, $\langle t_n \mid n \in \omega \rangle$, of elements of T s.t. t is dominated by some $f \in {}^\alpha\alpha \cap L$ where $t = \bigcup_n t_n$ and $\alpha = dom(t)$.

[proof]

Let $B := \{\xi < \kappa \mid \xi = Skull^{H(\kappa^+)}(\xi \cup \{\kappa, T\}) \cap \kappa\}$. For each $\xi \in B$, let $N_\xi := Skull^{H(\kappa^+)}(\xi \cup \{\kappa, T\})$. Clearly B is closed unbounded in κ. Note that $\{\xi \in B \mid o.t.(B \cap \xi) = \xi\}$ is also closed unbounded in κ. For each $\xi \in B$, let ξ_+ be the successor element of ξ in B.

Because $0^\sharp$ does not exists, if ξ has cofinality ω and $\xi > \omega_2$ then η is singular in L. So there is $\eta \in B$ s.t. $o.t.(B \cap \eta) = \eta$, η has cofinality ω and η is singular in L. Let η be such ordinal and $D \in L$ be a cofinal subset of η of order type less than η. Then we can take an increasing sequence in B, $\langle \xi_n \mid n \in \omega \rangle$, s.t. $\langle \xi_n \mid n \in \omega \rangle$ is cofinal in η and for each $n \in \omega$, the interval $[\xi_n, (\xi_n)_+)$ contains no element of D.

We shall define $t_n,\ s_n \in ({}^{<\kappa}\kappa)^L \cap N_{(\xi_n)_+}$ by induction on $n \in \omega$. Let $s_0 := \emptyset$. Assume s_n was defined so that $s_n \in ({}^{<\kappa}\kappa)^L \cap N_{(\xi_n)_+}$. Let t_n be an element of $T \cap N_{(\xi_n)_+}$. Then by the definition of B, $dom(t_n) < (\xi_n)_+$ and $ran(t_n) \subseteq (\xi_N)_+$. Let s_{n+1} be the end extension of t_n s.t. $dom(s_{n+1}) = \xi_{n+1}$ and for each $\beta \in \xi_{n+1} \backslash dom(t_n)$, $s_{n+1}(\beta) = 0$. Clearly $s_{n+1} \in N_{(\xi_{n+1})_+}$.

Note that $t := \bigcup_n t_n \in {}^\eta\eta$. Finally let $f \in {}^\eta\eta$ be the function s.t. for each $\beta \in \eta$, $f(\beta) :=$ the least element of D larger than β. Then clearly $f \in L$ and by the construction of t, f dominates t. □

4 Forcing extension in which ${}^{<\kappa}\kappa$ of the ground model is not κ-distributive

In this section, using Theorem 3.3, we shall see examples of forcing which force $({}^{<\kappa}\kappa)$ of the ground model is not κ-distributive.

First we shall define weakly compact cardinals. There are several characterisations of weakly compactness. Here we use the following. The original definition of weakly compactness and that it is equivalent with the following characterisation can be seen in Kanamori [6]. Every weakly compact cardinal is inaccessible. (see Kanamori [6].)

Definition 4.1
$\kappa \in On$ is weakly compact iff for every $S \subseteq V_\kappa$, there is a structure $\langle X, \in, T\rangle$ s.t. X is transitive, $\kappa \in X$ and $\langle V_\kappa, \in, S\rangle \prec \langle X, \in, T\rangle$.

Note: If $0^\sharp$ exists then every $i \in I$ is weakly compact in L. Let $S \subseteq L_i$. (Note that if κ is inaccessible in L then $(V_\kappa)^L = L_\kappa$.) Let $S = t^L[\vec{j}, i, \vec{k}\,]$ where t is a term, and $\vec{j}$ and $\vec{k}$ are sequences in I s.t. $\vec{j} < i < \vec{k}$. By Lemma 2.2, we can assume $min(\vec{k})$ is large enough. Let $i' \in I$ be s.t. $i < i' < \vec{k}$ and let $S' := t^L[\vec{j}, i', \vec{k}\,]$. Then, using the indiscernibility of I and Lemma 2.2, it can be easily seen that $\langle L_i, \in, S\rangle \prec \langle L_{i'}, \in, S'\rangle$.

Example 1

Let κ be a weakly compact cardinal in L and $\lambda < \kappa$ be a regular cardinal in L.

Stanley showed in Stanley [8] that there is a forcing over L which forces that $({}^{<\kappa}\kappa)^L$ is not κ-distributive. Moreover he showed the following. (For each forcing notion $\mathbb{Q}$, let $o(\mathbb{Q})$ be the least cardinal β s.t. $\mathbb{Q}$ adds a subset of β. $\mathbb{Q}$ is called uniform iff for every $p \in \mathbb{Q}$, $o(\mathbb{Q}) = o(\mathbb{Q}\restriction p)$, where $\mathbb{Q}\restriction p := \{q \in \mathbb{Q} \mid q \leq p\}$.)

" Assume κ is weakly compact in L. Then there is a forcing $\mathbb{S}$ over L s.t.
if $\mathbb{P} \in L$ is uniform in L and $o(\mathbb{P})^L = \kappa$ then $\Vdash_{\mathbb{S}}$ " $\mathbb{P}$ collapses κ ". "

But the forcing constructed in Stanley [8] is not aimed to adds C of Theorem 3.3. Here, in view of adding C of Theorem 3.3, we construct a forcing which forces that $({}^{<\kappa}\kappa)^L$ is not κ-distributive. As mentioned in the introduction, C of Theorem 3.3 can be also added using the forcing developed in Gitik-Magidor-Woodin [7]. The forcing developed there was something like Radin Forcing. But here we use a reverse Easton forcing and the forcing we construct gives another proof of Theorem 1 of Gitik-Magidor-Woodin [7].

Theorem 4.2
Assume κ is weakly compact in L and $\lambda < \kappa$ is regular in L. Then there is a forcing notion $\mathbb{P}$ satisfying the following.

If G is a $(L, \mathbb{P})$-generic filter then, in $L[G]$, κ and λ are regular, and there is $C \subseteq \kappa$ s.t.

1. *C is club in κ.*
2. *If $\alpha \in Lim(C)$ has cofinality λ in N then α is regular in L and $C \cap \alpha$ is faster than every constructible club subset of α.*

So $\Vdash_{\mathbb{P}}$ " $({}^{<\kappa}\kappa)^L$ is not λ^+-distributive ".

The forcing notion $\mathbb{P}$ which we construct is a three step iteration $\mathbb{S} * \mathbb{T} * \mathbb{U}$. Because of the second property, $\mathbb{P}$ must change cofinalities. Here $\mathbb{T}$ is a Levy collapse and κ becomes a successor cardinal in $L[G]$.

Forcing $\mathbb{S}$

$\mathbb{S}$ is a $(\kappa+1)$-stage reverse Easton iteration. For each regular $\alpha < \kappa$, $\mathbb{S}$ adds a club subset of α which is faster than every constructible club subset of α. First we introduce a forcing notion $FC(\alpha)$ which adds a fast club.

Let α be a regular cardinal. Then $FC(\alpha)$ consists of all pairs (t, F) s.t.

1. t is closed and bounded in α.
2. F is club in α.

For (t_1, F_1) and (t_2, F_2) in $FC(\alpha)$, $(t_1, F_1) \leq (t_2, F_2)$ iff

1. t_1 is an end extension of t_2.
2. $t_1 \backslash t_2 \subseteq F_2$.

For $FC(\alpha)$-generic G, let $C_G := \bigcup\{t \mid \exists F((t, F) \in G)\}$. C_G is club in α.

Lemma 4.3

Let α be regular and G be $FC(\alpha)$ generic. Then

1. *$FC(\alpha)$ is α-closed.*
2. *If $\alpha^{<\alpha} = \alpha$ then $FC(\alpha)$ has α^+-c.c..*
3. *C_G is faster than every club subset of α which is in the ground model.*
4. *$G = \{(t, F) \in FC(\alpha) \mid t \subseteq C_G \ \wedge \ C_G \backslash t \subseteq F\}$. So G can be recovered from C_G.*

[proof]

1. Assume $\langle (t_\xi, F_\xi) \mid \xi < \eta \rangle$ is a decreasing sequence of length $\eta < \alpha$. Let $t := (\bigcup_{\xi<\eta} t_\xi) \cup (sup(\bigcup_{\xi<\eta} t_\xi))$ and $F := \bigcap_{\xi<\eta} F_\xi$. Then it is easy to see that $(t, F) \leq (t_\xi, F_\xi)$ for each $\xi < \eta$. $\square$

2. Let $\{(t_\xi, F_\xi) \mid \xi < \alpha^+\} \subseteq FC(\alpha)$. Because $\alpha^{<\alpha} = \alpha$, there are $\xi < \eta$ s.t. $t_\xi = t_\eta$. Then $(t_\xi, F_\xi \cap F_\eta)$ is a common extension of (t_ξ, F_ξ) and (t_η, F_η). $\square$

3. Assume $C \in V$ is closed unbounded in α and is in the ground model. Then clearly $D_C := \{(t, F) \mid F \subseteq C\}$ is dense in $FC(\alpha)$. Assume $(t, F) \in D_C \cap G$. Then $C_G \backslash t \subseteq F$. So $C_G \backslash t \subseteq C$. $\square$

4. Let $H := \{(t, F) \in FC(\alpha) \mid t \subseteq C_G \ \wedge \ F \backslash t \subseteq C_G\}$. $H \subseteq G$ is trivial. We show $G \supseteq H$. Assume $(t, F) \in H \backslash G$. Then there is $(s, E) \in G$ s.t. (s, E) is incompatible with (t, F). Note that $t \neq s$ and both t and s are initial segments of C_G. First assume $t \subsetneq s$. If $s \backslash t \subseteq F$ then $(s, E \cap F)$ becomes a common extension of (t, F) and (s, E). So $s \backslash t \not\subseteq F$. But this contradicts to that $s \backslash t \subseteq C_G \backslash t \subseteq F$. Next assume $s \subsetneq t$. As above, we can show $t \backslash s \not\subseteq E$. So $t \backslash s \subseteq C_G \backslash s \not\subseteq E$. This contradicts to $(s, E) \in G$. $\square$

Now we shall define $\mathbb{S}$ in L.

Let $\langle (\mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi) \mid \xi \leq \kappa \rangle$ be a forcing iteration s.t.

- If ξ is regular then $\Vdash_\xi$ " $\dot{\mathbb{Q}}_\xi = FC(\xi)$ ". Otherwise $\Vdash_\xi$ " $\dot{\mathbb{Q}}_\xi$ is a trivial forcing ".
- If ξ is regular then $\mathbb{P}_\xi$ is the direct limit of $\langle \mathbb{P}_\eta \mid \eta < \xi \rangle$ and if ξ is singular then the inverse limit.

Then let $\mathbb{S} := \mathbb{P}_{\kappa+1}$.

Note: There are two ways in defining $\mathbb{P}_\alpha$. One is to define $\mathbb{P}_\alpha$ as a set consisting of total functions p on α s.t. for each $\xi < \alpha$, $p(\xi)$ is a $\mathbb{P}_\xi$-name of an element of $\dot{\mathbb{Q}}_\xi$. Another is to define $\mathbb{P}_\alpha$ as a set consisting of partial functions p on α s.t. $supp(p) = dom(p)$. In this paper, we use the latter definition. So, for each $\eta < \xi$, $\mathbb{P}_\xi \subseteq \mathbb{P}_\eta$ and if $\mathbb{P}_\alpha$ is the direct limit of $\langle \mathbb{P}_\xi \mid \xi < \alpha \rangle$ then $\mathbb{P}_\alpha = \bigcup_{\xi<\alpha} \mathbb{P}_\xi$.

We shall examine this iteration. First we see basic properties.

Lemma 4.4

1. *Assume $\eta \leq \kappa$. Then $\mathbb{P}_{\eta+1} \subseteq L_{\eta^+}$. So if $\alpha \leq \kappa$ is regular then $\mathbb{P}_\alpha \subseteq L_\alpha$.*
2. *Assume $\xi < \eta$. Then $\Vdash_\xi$ " $\dot{\mathbb{P}}_{\xi,\eta}$ is ξ-closed ", where $\dot{\mathbb{P}}_{\xi,\eta}$ is the canonical $\mathbb{P}_\xi$-name s.t. $\mathbb{P}_\xi * \dot{\mathbb{P}}_{\xi,\eta} = \mathbb{P}_\eta$.*
3. *Assume $\eta \leq \kappa$ is regular. Then $\mathbb{P}_{\eta+1}$ has η^+-c.c..*
4. *Assume $\eta \leq \kappa + 1$. Then $\mathbb{P}_\eta$ preserves cofinalities.*
5. *$\mathbb{S}$ preserves GCH.*

[proof]
1. It suffices to show the first statement for regular η. Other cases can be shown easily from this. We show by induction on η. Assume for every regular $\xi < \eta$, $\mathbb{P}_\xi \subseteq L_{\xi^+}$. Then $\mathbb{P}_\eta \subseteq L_\eta$. It suffices to show that for every $\mathbb{P}_\eta$-name $\dot{q}$ of an element of $\dot{\mathbb{Q}}_\eta$, there is $\dot{r} \in L_{\eta^+}$ s.t. $\Vdash_\eta$ " $\dot{r} = \dot{q}$ ". Let $\dot{t}_q$ and $\dot{F}_q$ be $\mathbb{P}_\eta$-names s.t. $\Vdash_\eta$ " $\dot{q} = (\dot{t}_q, \dot{F}_q)$ ". Then let $\dot{t}_r$ be a set consisting of all pairs $(\check{\alpha}, p)$ s.t. $\alpha < \eta$, $p \in \mathbb{P}_\eta$ and $p \Vdash_\eta$ " $\alpha \in \dot{t}_q$ ". Define $\dot{F}_r$ similarly. Then it can be easily seen that $\dot{t}_r$ and $\dot{F}_r$ become $\mathbb{P}_\eta$-names s.t. $\Vdash_\eta$ " $\dot{t}_r = \dot{t}_q \ \wedge \ \dot{F}_r = \dot{F}_q$ ". Moreover, because $\mathbb{P}_\eta \subseteq L_\eta$, $\dot{t}_r$ and $\dot{F}_r$ are in L_{η^+}. So $\dot{r} := (\dot{t}_r, \dot{F}_r) \in L_{\eta^+}$ and $\Vdash_\eta$ " $\dot{q} = \dot{r}$ ". □

2. If ξ is not regular then a trivial forcing is iterated at the ξ-th stage. So it suffices to show for regular ξ. By 1, $\mathbb{P}_\xi \subseteq L_\xi$ and so $\mathbb{P}_\xi$ has ξ^+-c.c.. Then $\Vdash_\xi$ " $\dot{\mathbb{P}}_{\xi,\eta}$ is a iteration of ξ-closed forcing notions and if δ is a limit ordinal s.t. $cf(\delta) < \xi$ then inverse limit is taken at δ ". (see Theorem 5.4 of Baumgartner [2].) So $\Vdash_\xi$ " $\dot{\mathbb{P}}_{\xi,\eta}$ is ξ-closed ". □

3 and 4. We shall show 3 and 4 by induction on η simultaneously.
Case 1: η is a singular cardinal.

It suffices to show only 4. Assume there are regular $\alpha < \beta$ s.t. $\Vdash_\eta$ " $cf(\beta) = \alpha$. Because $\mathbb{P}_\eta \subseteq L_{\eta^+}$, $\mathbb{P}_\eta$ has η^+-c.c.. So $\alpha < \eta$. Then by 2, $\Vdash_{\alpha+1}$ " $cf(\beta) = \alpha$ ". This contradicts to the induction hypothesis.
Case 2: η is regular.
First we show 4. By 1, $\mathbb{P}_\eta \subseteq L_\eta$ and $\mathbb{P}_\eta$ has η^+-c.c.. So $\mathbb{P}_\eta$ preserves cofinalities above η. But as in the proof of the previous case, we can show $\mathbb{P}_\eta$ preserves cofinalities $\leq \eta$.

Next we show 3. Because $\mathbb{P}_\eta$ preserves regularity of η and $\mathbb{P}_\eta$ has η^+-c.c., by 2 of Lemma 4.3, it suffices to show $\Vdash_\eta$ " $\eta^{<\eta} = \eta$ ". Assume $\alpha < \alpha^+ < \eta$. Because $\mathbb{P}_{\alpha+1} \subseteq L_{\alpha^+}$, there are at most $(\eta^{\alpha^+})^\alpha$ nice names of elements of η^α. So $\Vdash_{\alpha+1} \eta^\alpha = \eta$ ". Moreover $\Vdash_{\alpha+1}$ " $\dot{\mathbb{P}}_{\alpha+1,\eta}$ is α^+-closed ". So $\Vdash_\eta$ " $\eta^\alpha = \alpha$. Next assume $\alpha^+ = \eta$. Then by induction hypothesis, $\mathbb{P}_{\alpha+1}$ has η-c.c. and $|\mathbb{P}| \leq \eta$. So $\Vdash_{\alpha+1}$ " $\eta^\alpha = \eta$ ". (Count nice names.) Because trivial forcing are iterated at the interval $[\alpha + 1, \eta)$, $\Vdash_\eta$ " $\eta^\alpha = \eta$ ".
Case 3: Otherwise.

Clear by the induction hypothesis.

5. Let η be a L-cardinal. It suffices to show $\Vdash_{\eta+1}$ " $2^\eta = \eta^+$ ". Let G be a $(L, \mathbb{P}_{\eta+1})$-generic filter and $\sigma^G \subseteq \eta$. Because $\mathbb{P}_{\eta+1} \subseteq L_{\eta^+}$, there is a surjection $f \in L$ from η^+ to $\mathbb{P}_{\eta+1}$. Then, because $\mathbb{P}_{\eta+1}$ does not collapse η^+, there is $\xi_\sigma < \eta^+$ s.t. for all $\alpha < \eta$,

$\alpha \in \sigma^G \iff \exists\delta < \xi_\sigma(f(\delta) \in G \ \wedge\ f(\delta) \Vdash \text{“ } \alpha \in \sigma \text{ ”})$. Let $S^\alpha_\sigma := \{\delta < \xi_\sigma \mid f(\delta) \Vdash \text{“ } \alpha \in \sigma \text{ ”}\}$. Then σ_G is uniquely determined by ξ_σ and $\langle S^\alpha_\sigma \mid \alpha < \eta\rangle$. But there are at most η^+-many such pairs. So, in $M[G]$, $2^\eta = \eta^+$.

□

Next we show the key lemma. For regular η and $\mathbb{P}_{\eta+1}$-generic filter G, let

$A_G := \{\xi < \eta \mid \xi \text{ is regular and } C_{H_\xi} \text{ is an initial segment of } C_{H_\eta}\}$,

where for each $\xi \leq \eta$, $G_\xi := G \cap \mathbb{P}_\xi$ and H_ξ is the $(\dot{\mathbb{Q}}_\xi)^{G_\xi}$-generic filter obtained naturally from G.

Lemma 4.5
Assume G is a $(L, \mathbb{P}_{\kappa+1})$-generic filter. Then A_G is stationary in κ in $L[G]$.

To show this we need a lemma about weakly compact cardinal.

Lemma 4.6
Assume κ is weakly compact and $\langle M, \in\rangle$ is a transitive structure s.t. $\kappa \in M$, $(2^\kappa)^M = \kappa$ and M satisfies one of the following conditions.

1. *$M \models ZFC$.*
2. *$M \models ZF^-$ and there is a well order $<_M$ of M which is definable over M.*

Then there is a transitive N and a elementary embedding $j : M \to N$ s.t. $crit(j) = \kappa$.

[proof]

Let M and κ be as in the lemma. Let $f : \kappa \to \mathcal{P}(\kappa)^M$ be a bijection and $S \subseteq \kappa \times \kappa$ be s.t. $(\xi, \eta) \in S$ iff $\eta \in f(\xi)$. Because κ is weakly compact and $S \subseteq V_\kappa$, there is a transitive structure $\langle X, \in, T\rangle$ s.t. $\kappa \in X$ and $\langle V_\kappa, \in, S\rangle \prec \langle X, \in, T\rangle$. Then let $U \subseteq \mathcal{P}(\kappa)^M$ be s.t. $f(\xi) \in U$ iff $(\xi, \kappa) \in T$. We show that U is an κ-complete ultrafilter. We show only κ-completeness. Others can be shown similarly. Assume $B \subseteq \kappa$, $|B| < \kappa$ and $\forall\xi \in B(f(\xi) \in U)$. Let $\eta < \kappa$ be s.t. $f(\eta) = \bigcap_{\xi \in B} f(\xi)$. Then $\langle V_\kappa, \in, S\rangle \models$ “ If $(\xi, \alpha) \in S$ for every $\xi \in B$, then $(\eta, \alpha) \in S$ ”. Because for every $\xi \in B$ $(\xi, \kappa) \in T$, so by the elementarity $(\eta, \kappa) \in T$ and $f(\eta) \in U$.

Let N be ${}^\kappa M \cap M/U$, the ultraproduct of M by U and $j : M \to N$ be the canonical embedding. Then $crit(j) = \kappa$. Because of the assumption 1 or 2 of M, Los theorem can be applied. So j is an elementary embedding. □

Note: Let M, κ, N and j be as above. Assume $a \in H(\kappa^+)^M$. Then there is $x \in \mathcal{P}(\kappa) \cap M$ s.t. x codes a. Because $x = j(x) \cap \lambda \in N$, so $a \in N$. So $H(\kappa^+)^M \subseteq H(\kappa^+)^N$.

Remark: It follows from Lemma 4.6 that if $0^\sharp$ does not exist then for every weakly compact λ, $\lambda^+ = (\lambda^+)^L$.

[proof of lemma 4.5]

Let $p \in \mathbb{P}_{\kappa+1}$ and $\dot{C}$ be a $\mathbb{P}_{\kappa+1}$-name of a closed unbounded subset of κ. We show that $\neg\, p \Vdash$ “ $\dot{C} \cap A_{\dot{G}} = \emptyset$ ” where $\dot{G}$ is a $\mathbb{P}_{\kappa+1}$-name of its generic filter.

Let δ be a sufficiently large regular cardinal. Let $X := Skull^{L_\delta}(\kappa \cup \{\kappa, p, \dot{C}\})$ and $\pi : X \to M$ be the transitive collapse. Note that $\pi \restriction X \cap L_{\kappa^+} = id \restriction X \cap L_{\kappa^+}$, because $X \cap L_{\kappa^+}$ is transitive. Note also that we can assume p and $\dot{C}$ are in L_{κ^+}, because $\mathbb{P}_{\kappa+1} \subseteq L_{\kappa^+}$ and $\mathbb{P}_{\kappa+1}$ has κ^+-c.c.. So $\pi(p) = p$ and $\pi(\dot{C}) = \dot{C}$. Let $\langle(\mathbb{P}^M_\xi, \dot{\mathbb{Q}}^M_\xi) \mid \xi \leq \kappa\rangle := \pi(\langle(\mathbb{P}_\xi, \dot{\mathbb{Q}}_\xi) \mid \xi \leq \kappa\rangle)$ and $\mathbb{P}^M_{\kappa+1} := \pi(\mathbb{P}_{\kappa+1})$. Then for every $\xi \leq \kappa$, $\mathbb{P}^M_\xi = \mathbb{P}_\xi$ and $\mathbb{P}^M_{\kappa+1} = \mathbb{P}_{\kappa+1} \cap M$.

Because $\kappa \in M$, $|M| = \kappa$, and M satisfies the condition 2 of Lemma 4.6, there is a transitive N and an elementary embedding $j : M \to N$ s.t. $crit(j) = \kappa$. Let

- $\gamma := j(\kappa)$
- $\mathbb{P}^N_{\gamma+1} := j(\mathbb{P}^M_{\kappa+1})$
- $\langle(\mathbb{P}^N_\xi, \dot{\mathbb{Q}}^N_\xi) \mid \xi < \gamma\rangle := j(\langle(\mathbb{P}^M_\xi, \dot{\mathbb{Q}}^M_\xi) \mid \xi < \kappa\rangle)$.

Then, in N, $\langle(\mathbb{P}^N_\xi, \dot{\mathbb{Q}}^N_\xi) \mid \xi \le \gamma\rangle$ is a iteration of FC and $\mathbb{P}^N_{\gamma+1} = \mathbb{P}^N_\gamma * \dot{\mathbb{Q}}^N_\gamma$. Because $crit(j) = \kappa$, $\langle(\mathbb{P}^N_\xi, \dot{\mathbb{Q}}^N_\xi) \mid \xi < \kappa\rangle = \langle(\mathbb{P}^M_\xi, \dot{\mathbb{Q}}^M_\xi) \mid \xi < \kappa\rangle = \langle(\mathbb{P}_\xi, \mathbb{Q}_\xi) \mid \xi < \kappa\rangle$. Then, because $\mathbb{P}_\kappa$ is the direct limit, $\mathbb{P}^N_\kappa = \mathbb{P}^M_\kappa = \mathbb{P}_\kappa$. So $\mathbb{P}^N_{\kappa+1} = \mathbb{P}_{\kappa+1} \cap N$, because the statement " $\dot{q}$ is a $\mathbb{P}_\kappa$-name of an element of $FC(\kappa)$ " is absolute between N and L. Moreover $\mathbb{P}^M_{\kappa+1} \subseteq \mathbb{P}^N_{\kappa+1}$, for $L_{(\kappa^+)^M} \subseteq L_{(\kappa^+)^N}$.

Because $\pi(p) = p$, $\pi(\dot{C}) = \dot{C}$, and j is elementary, it suffices to show

$$(*)\quad N \models \text{“} \neg\, j(p) \Vdash_{\mathbb{P}^N_{\gamma+1}} \text{“} j(\dot{C}) \cap A_{\dot{G}} = \emptyset \text{”} \text{”},$$

where $\dot{G}$ is the $\mathbb{P}^N_{\gamma+1}$-name of its generic filter.

Let G_γ be a $(N, \mathbb{P}^N_\gamma)$-generic filter s.t. $p \in G_\gamma$. (Note that $p \in \mathbb{P}^M_{\kappa+1} \subseteq \mathbb{P}^N_{\kappa+1} \subseteq \mathbb{P}^N_\gamma$.) For each $\xi \le \gamma$, let

- $G_\xi := G_\gamma \cap \mathbb{P}^N_\xi$
- $Q^N_\xi := (\dot{\mathbb{Q}}^N_\xi)^{G_\xi}$
- $H_\xi :=$ the $(N[G_\xi], Q^N_\xi)$-generic filter obtained from G_γ. (If $\xi < \gamma$.)

Claim $G_{\kappa+1} \cap \mathbb{P}^M_{\kappa+1}$ is $(M, \mathbb{P}^M_{\kappa+1})$-generic.
[proof of claim]

It suffices to show if $D \in M$ is a maximal antichain in $\mathbb{P}^M_{\kappa+1}$ then $D \in N$ and D is a maximal antichain in $\mathbb{P}^N_{\kappa+1}$. Assume D is a maximal antichain in $\mathbb{P}^M_{\kappa+1}$ and $D \in M$. Because $M \models$ " $\mathbb{P}^M_{\kappa+1}$ has κ^+-c.c. ", $D \in L_{(\kappa^+)^M}$. So $\pi^{-1}(D) = D$. Then, by the elementarity of π^{-1}, D is a maximal antichain in $\mathbb{P}_{\kappa+1}$. Note that, for each p_1 and p_2 in $\mathbb{P}^N_{\kappa+1}$, $N \models$ " p_1 and p_2 are compatible in $\mathbb{P}^N_{\kappa+1}$ " iff $L \models$ " p_1 and p_2 are compatible in $\mathbb{P}_{\kappa+1}$ ". This is because $\mathbb{P}_\kappa = \mathbb{P}^N_\kappa \in N$ and compatibility of elements of $FC(\kappa)$ can be written in Σ_0-formula. So D is a maximal antichain in $\mathbb{P}^N_{\kappa+1}$. $D \in N$ follows from $D \in L_{(\kappa^+)^M}$. □claim

By the claim above, we see also that G_κ is $(M, \mathbb{P}^M_\kappa)$-generic. Let $Q^M_\kappa := (\dot{\mathbb{Q}}^M_\kappa)^{G_\kappa}$. Let $j^* : M[G_\kappa] \to N[G_\gamma]$ be a function s.t. for each $\mathbb{P}^M_\kappa$-name $\dot{a}$, $j^*(\dot{a}^{G_\kappa}) = j(\dot{a})^{G_\gamma}$. Note that $\mathbb{P}^M_\kappa \subseteq L_\kappa$ and so $j \restriction \mathbb{P}^M_\kappa = id \restriction \mathbb{P}^M_\kappa$. So for each $q \in \mathbb{P}^M_\kappa$, $q \in G_\kappa$ iff $j(q) \in G_\kappa$. So j^* is an elementary embedding extending j.

Let $\dot{B} \in M[G_\kappa]$ be the Q^M_κ-name obtained naturally from $\dot{C}$ and G_κ s.t. for every $(M[G_\kappa], Q^M_\kappa)$-generic H, $\dot{C}^{G_\kappa * H} = \dot{B}^H$. Then $j^*(\dot{B})$ becomes Q^N_γ-name s.t. for every $(N[G_\gamma], Q^N_\gamma)$-generic H, $j(\dot{C})^{G_\gamma * H} = j^*(\dot{B})^H$.

We shall find $(t, F) \in Q^N_\gamma$ s.t.

- $(t, F) \le j(p)(\gamma)^{G_\gamma}$, and
- $(t, F) \Vdash_{Q^N_\gamma}$ " $C_{H_\kappa} = C_{\dot{H}}$ ",

where $\dot{H}$ is the Q^N_γ name of its generic filter. Assume such (t, F) exists. Let H_γ be a $(N[G_\gamma], Q^N_\gamma)$-generic filter s.t. $(t, F) \in H_\gamma$, and let $G := G_\gamma * H_\gamma$. Then $j(p) \in G$ and $\kappa \in j(\dot{C})^G \cap A_G$. So $(*)$ is true.

Work in $N[G_\gamma]$. Let $t := C_{H_\kappa} \cup \{\kappa\}$. So if $(t, F) \in H$ then C_{H_κ} is an initial segment of C_H. We shall define F.

Claim For every $\alpha < \kappa$, there are $\beta \in [\alpha, \kappa)$ and E s.t. $(t, E) \Vdash$ " $\beta \in j^*(\dot{B})$ ".
[proof of claim]

Let $\alpha < \kappa$. By the previous claim $G_{\kappa+1}$ is $(M, \mathbb{P}^M_{\kappa+1})$-generic. So $\dot{C}^{G_{\kappa+1}}$ is unbounded in

κ. Let $\beta < \kappa$ be the least element of $\dot{C}^{G_{\kappa+1}}$ above α. Then there is $(s, D) \in H_\kappa \cap Q^M_\kappa$ s.t. in $M[G_\kappa]$, $(s, D) \Vdash$ " $\beta \in \dot{B}$ ". So by the elementarity of j^*, $(s, j^*(D)) \Vdash$ " $\beta \in j^*(\dot{B})$ ". (Note that $j^*(s) = s$ and $j^*(\beta) = \beta$, because $crit(j^*) = \kappa$.) Note also that $j^*(D) \cap \kappa = D$. Let $E := j^*(D)$. Because $(s, D) \in H_\kappa$ and $E \cap \kappa = D$, $(t, E) \leq (s, E)$. So $(t, E) \Vdash$ " $\beta \in j^*(\dot{B})$ ". □claim

For each $\alpha < \kappa$, choose $E_\alpha \subseteq \gamma$ s.t. for some $\beta \in [\alpha, \kappa)$, $(t, E_\alpha) \Vdash$ " $\beta \in j^*(\dot{B})$ ".

Next assume $p(\kappa)^{G_\kappa} = (s_p, D_p)$. Then $j(p)(\gamma)^{G_\gamma} = j^*(p(\kappa)^{G_\kappa}) = (s_p, j^*(D_p))$. Let $E_p := j^*(D_p)$. Then $E_p \cap \kappa = D_p$. Moreover because $p \in G_{\kappa+1}$, $(s_p, D_p) \in H_\kappa$. So $(t, E_p) \leq (s_p, E_p) = j(p)(\gamma)^{G_\gamma}$.

Finally let $F := E_p \cap (\bigcap_{\alpha<\kappa} F_\alpha)$. Then $(t, F) \leq j(p)(\gamma)^{G_\gamma}$ and for every $\alpha < \kappa$, there is $\beta \in [\alpha, \kappa)$ s.t. $(t, F) \Vdash$ " $\beta \in j^*(\dot{B})$ ". Because $\Vdash$ " $j^*(\dot{B})$ is closed unbounded ", $(t, F) \Vdash$ " $\kappa \in j^*(\dot{B})$ ". This completes the proof. □lemma

Forcing $\mathbb{T}$

Let G be $(L, \mathbb{S})$-generic. We shall work in $L[G]$. Forcing $\mathbb{T}^G$ is Levy collapse $Col(\lambda, \kappa)$. Note that $L[G] \models$ " λ is regular ".

Let $S := \{\eta \mid \lambda < \eta < \kappa\}$. $\mathbb{T}^G$ consists of all functions f s.t. $dom(f) \subseteq S \times \lambda$, $|f| < \lambda$ and $\forall(\eta, \xi) \in S \times \lambda(f(\eta, \xi) \in \eta)$. For each $f,\ g \in \mathbb{T}^G$, $f \leq g$ iff $g \subseteq f$.

Assume J is $\mathbb{T}^G$-generic. For each $\eta \in S$, let $J_\eta : \lambda \to \eta$ be the function s.t. $J_\eta(\xi) = \alpha$ iff $\exists f \in J(f(\eta, \xi) = \alpha)$. Then an easy density argument shows that J_η is onto. So for each $\eta \in S$, $\Vdash_{\mathbb{T}^G}$ " $|\eta| \leq \lambda$ ". Moreover $\mathbb{T}^G$ is λ-closed. So $\mathbb{T}^G$ preserves regularity of λ and if $\eta \in S$ is regular in $L[G]$ then $\Vdash_{\mathbb{T}^G}$ " $cf(\eta) = \lambda$ ".

Note that, by lemma 4.4, κ is inaccessible in $L[G]$. So Δ-system lemma shows $\mathbb{T}^G$ has κ-c.c.. In particular, κ is regular in $L[G][J]$.

Recall that A_G is a stationary subset of κ. We show that $\mathbb{T}^G$ preserves stationarity of A_G.

Lemma 4.7

Assume α is regular and $\mathbb{B}$ is an α-c.c. forcing notion. Then $\mathbb{B}$ preserves stationarity of subsets of α.

[proof]

We show that if $\dot{C}$ is a $\mathbb{B}$-name of a closed unbounded subset of α then there is D in the ground model s.t. D is closed unbounded in α and $\Vdash_{\mathbb{B}}$ " $D \subseteq \dot{C}$. Clearly lemma follows from this.

Assume $\dot{C}$ is a $\mathbb{B}$-name of a closed unbounded subset of α. Let $D := \{\beta < \alpha \mid \Vdash_{\mathbb{B}}$ " $\beta \in \dot{C}$ "$\}$. Then clearly $\Vdash_{\mathbb{B}}$ " $D \subseteq \dot{C}$ " and D is closed subset of α. So it suffices to show that D is unbounded in α. Let $\xi < \alpha$. By induction on $n \in \omega$, define an increasing sequence of ordinals $\langle \xi_n \mid n \in \omega \rangle$. Let $\xi_0 := \xi$. Assume ξ_n is defined. Let

$$C_n := \{\eta < \alpha \mid \exists p \in \mathbb{B}\ (p \Vdash \text{“ } \eta \text{ is the least element of } \dot{C} \text{ above } \xi_n \text{ ”})\}$$

Because $\mathbb{B}$ has α-c.c. $|C_n| < \alpha$. Let $\xi_{n+1} := supC_n$. Note that $\Vdash$ " $\dot{C} \cap (\xi_n, \xi_{n+1}] \neq \emptyset$ ".

Let $\eta := sup_{n\in\omega}\xi_n$. Because $\dot{C}$ is forced to be closed unbounded, $\Vdash$ " $\eta \in \dot{C}$ ". By the construction, $\xi < \eta < \alpha$. □

Forcing $\mathbb{U}$

$\mathbb{U}$ is club shooting. Let G be $(L, \mathbb{S})$-generic and J be $(L[G], \mathbb{T}^G)$-generic. Then $L[G][J] \models$ " $\kappa = \lambda^+$ and A_G is stationary in κ ". We shall work in $L[G][J]$. $\mathbb{U}$ adds a closed unbounded subset $C \subseteq \kappa$ s.t. if $\alpha \in Lim(C)$ and $cf(\alpha) = \lambda$ then $\alpha \in A_G$.

Let $S := \{\xi < \kappa \mid cf(\xi) < \lambda\} \cup A_G$. Then let

$\mathbb{U}^{G*J} := \{t \subseteq S \mid t \text{ is a closed bounded subset of } \kappa\}$,

and for each $s,\ t \in \mathbb{U}^{G*J}$, $s \leq t$ iff s is an end extension of t.

We see that $\mathbb{U}^{G*J}$ preserves cofinalities. We use the following well known theorem. See Abraham-Shelah [1].

Definition 4.8
Assume α is regular and $S \subseteq \alpha$. Then S is fat iff for every closed unbounded $C \subseteq \alpha$ and regular $\beta < \alpha$, there is $t \subseteq C \cap S$ s.t. t is closed and the order type of t is $\beta + 1$.

Theorem 4.9 (Abraham-Shelah [1])
Assume α is either an inaccessible cardinal or the successor of a regular cardinal β s.t. $\beta^{<\beta}$. Assume $S \subseteq \alpha$ is fat. Let P be a forcing notion s.t. P consists of all closed bounded subset of S and for each $p,\ q \in P$, $p \leq q$ iff p is an end extension of q. Then P is α-distributive.

Because $\mathbb{S}$ preserves GCH and $\mathbb{T}$ is λ-closed, $\lambda^{<\lambda} = \lambda$ in $L[G][J]$. So to apply above theorem, it suffices to show that S is fat. Assume $C \subseteq \kappa$ is a closed unbounded. Because $\kappa = \lambda^+$, it suffices to find $t \subseteq S \cap C$ s.t. $o.t.(t) = \lambda + 1$. Because A_G is stationary, there is $\eta \in Lim(C) \cap A_G$. Let $D \subseteq \eta$ be a closed unbounded subset of η s.t. $o.t.(D) = \lambda$. Then $t := \{\eta\} \cup (D \cap C) \subseteq C \cap S$ and $o.t.(t) = \lambda + 1$.

Proof of Theorem 4.2

We show that $\mathbb{P} = \mathbb{S} * \mathbb{T} * \mathbb{U}$ adds an closed unbounded subset of κ which satisfies the properties of Theorem 3.3. Let G be $(L, \mathbb{S})$-generic, J be $(L[G], \mathbb{T}^G)$-generic and K be $(L[G][J], \mathbb{U}^{G*J})$-generic. Then $\bigcup K$ is closed unbounded in κ and if $\alpha \in Lim(\bigcup K)$ and $cf(\alpha) = \lambda$ then $\alpha \in A_G$. (Note that $\mathbb{U}$ does not change cofinalities.) Let $C := C_{H_\kappa} \cap (\bigcup K)$. (Here H_κ is the $(L[G \cap \mathbb{P}_\kappa], (\dot{\mathbb{Q}}_\kappa)^{G \cap \mathbb{P}_\kappa})$-generic filter obtained from G. See the proof of Lemma 4.4.) If $\alpha \in Lim(C)$ has cofinality λ then $\alpha \in A_G$. So α is regular in L. Moreover $C \cap \alpha \subseteq C_{H_\alpha}$, so $C \cap \alpha$ is faster than every constructible closed unbounded subset of α.

$\square$theorem

Example 2

In 1, we gave the forcing $\mathbb{P}$ s.t. $\Vdash_{\mathbb{P}}$ " $({}^{<\kappa}\kappa)^L$ is not λ^+-distributive ". First we show that $\Vdash_{\mathbb{P}}$ " $({}^{<\kappa}\kappa)^L$ is λ-distributive ". Note that, because $\Vdash_{\mathbb{P}}$ " $\kappa = \lambda^+$ ", this implies that $\Vdash_{\mathbb{P}}$ " $({}^{<\kappa}\kappa)^L$ is η-distributive " for every $\eta < \kappa$.

Lemma 4.10
Let κ be a regular cardinal in V and $\mathbb{P}$, $\mathbb{Q}$ be partial orders in V.

1. *Assume $\mathbb{P}$ has κ-c.c. and $\mathbb{Q}$ is κ-closed. Then $\Vdash_{\mathbb{P}}$ " $\check{\mathbb{Q}}$ is κ-distributive ". In particular, $\Vdash_{\mathbb{P}}$ " $({}^{<\kappa}\kappa)^V$ is κ-distributive ".*
2. *Assume $\mathbb{P}$ is κ-closed and $\mathbb{Q}$ is κ-distributive, then $\Vdash_{\mathbb{P}}$ " $\check{\mathbb{Q}}$ is κ-distributive ".*

[proof]
1. We shall show

($*$) Assume $\dot{D}$ is a $\mathbb{P}$-name s.t. $\Vdash_{\mathbb{P}}$ " $\dot{D}$ is a dense open subset of $\check{\mathbb{Q}}$. Then there is $E \in V$ s.t. E is dense open in $\mathbb{Q}$ and $\Vdash_{\mathbb{P}}$ " $\check{E} \subseteq \dot{D}$ ".

First, assuming ($*$), we show 1. Let $p \in \mathbb{P}$ and $\dot{X} \in V^{\mathbb{P}}$ be s.t. $p \Vdash$ " $\forall D \in \dot{X}(D$ is dense open in $\mathbb{Q}) \wedge |\mathbb{Q}| < \kappa$ ". It suffices to find $q \leq_{\mathbb{P}} p$ s.t. $q \Vdash$ " $\bigcap \dot{X}$ is dense open in $\mathbb{Q}$ ".

Because $\mathbb{P}$ has κ-c.c. there is $q \leq_{\mathbb{P}} p$ and $\lambda < \kappa$ s.t. $q \Vdash$ " $|\dot{X}| = \lambda$ ". Take a sequence of $\mathbb{P}$-names $\langle \dot{D}_\xi \mid \xi < \lambda \rangle$ s.t. $q \Vdash$ " $\dot{X} = \{\dot{D}_\xi \mid \xi < \lambda\}$ ". By $(*)$ we can take $E_\xi \in V$ s.t. E_ξ is dense open in $\mathbb{Q}$ and $q \Vdash$ " $E_\xi \subseteq \dot{D}_\xi$ " for each $\xi < \lambda$. Then by the κ-closedness of $\mathbb{Q}$, $E := \bigcap_{\xi<\lambda} E_\xi$ is dense open in $\mathbb{Q}$. Moreover $q \Vdash$ " $E \subseteq \bigcap \dot{X}$ ". So $q \Vdash$ " $\bigcap \dot{X}$ is dense open in $\mathbb{Q}$ ".

Now we shall show $(*)$. Let $E := \{q \in \mathbb{Q} \mid \Vdash_{\mathbb{P}}$ " $q \in \dot{D}$ "$\}$. Then it is clear that E is open in $\mathbb{Q}$ and $\Vdash_{\mathbb{P}}$ " $E \subseteq \dot{D}$ ". So it suffices to show E is dense in $\mathbb{Q}$.

Take an arbitrary $q \in \mathbb{Q}$. We shall find $r \leq_{\mathbb{Q}} q$ s.t. $r \in E$. By induction on ξ, take $(p_\xi, q_\xi) \in \mathbb{P} \times \mathbb{Q}$ so that $\langle p_\xi \rangle_\xi$ becomes an antichain in $\mathbb{P}$ and $\langle q_\xi \rangle_\xi$ becomes a descending chain in $\mathbb{Q}$.
We stop the induction when $\{p_\eta \mid \eta < \xi\}$ becomes a maximal antichain. So this induction stops in less than κ stages.

Fix a well order $\lhd$ of $\mathbb{P} \times \mathbb{Q}$. Let (p_0, q_0) be the $\lhd$-least element of $\mathbb{P} \times \mathbb{Q}$ s.t. $q_0 \leq_{\mathbb{Q}} q$ and $p_0 \Vdash$ " $q_0 \in \dot{D}$ ".

Assume (p_η, q_η), $\eta < \xi$ are taken and $\{p_\eta \mid \eta < \xi\}$ is not maximal. Then let (p_ξ, q_ξ) be the $\lhd$-least element of $\mathbb{P} \times \mathbb{Q}$ s.t.

- $q_\xi < q_\eta$ for each $\eta < \xi$.
- p_ξ is incompatible with p_ξ for each $\eta < \xi$
- $p_\xi \Vdash$ " $q_\xi \in \dot{D}$ ".

Note that there is such (p_ξ, q_ξ) because of κ-closedness of $\mathbb{Q}$.

Let $\alpha < \kappa$ be s.t. this induction stops at the α-th stage. Let $r \in \mathbb{Q}$ be s.t. $r \leq_{\mathbb{Q}} q_\xi$ for each $\xi < \alpha$. Then for each $\xi < \alpha$, $p_\xi \Vdash$ " $q \in \dot{D}$ ". Because $\langle p_\xi \mid \xi < \alpha \rangle$ is a maximal antichain, so $r \in E$. By the construction, it is clear $r \leq_{\mathbb{Q}} q$. □

2. Because $\mathbb{Q}$ adds no sequence of elements of $\mathbb{P}$ of length less than κ, $\Vdash_{\mathbb{Q}}$ " $\mathbb{P}$ is κ-closed ". So $\mathbb{P} \times \mathbb{Q}$ is κ distributive. Because 2-step iteration of κ-distributive forcing is κ-distributive, $\mathbb{P} \times \mathbb{Q}$ is κ-distributive. Note that if $\Vdash_{\mathbb{P}}$ " $\mathbb{Q}$ is not κ distributive " then $\mathbb{P} \times \mathbb{Q}$ is not κ-distributive. So $\Vdash_{\mathbb{P}}$ " $\mathbb{Q}$ is κ-distributive ". □

Now we show that $\Vdash_{\mathbb{P}}$ " $({}^{<\kappa}\kappa)^L$) is λ-distributive ". It suffices to show that if $\eta < \lambda$ is regular in L then $\Vdash_{\mathbb{P}}$ " $({}^{<\kappa}\kappa)^L$ is η^+-distributive ". Let $\eta < \lambda$ be regular in L. Then $\mathbb{S}$ divides into $\mathbb{P}_{\eta+1} * \dot{\mathbb{P}}_{\eta+1,\kappa+1}$. By 3 of Lemma 4.4, $\mathbb{P}_{\eta+1}$ has η^+-c.c., and by 2 of Lemma 4.4, $\Vdash_{\mathbb{P}_{\eta+1}}$ " $\dot{\mathbb{P}}_{\eta+1,\kappa+1}$ is η^+-closed ". It can be easily seen that $\Vdash_{\mathbb{S}}$ " $\mathbb{T}$ is λ-closed " and $\Vdash_{\mathbb{S}*\mathbb{T}}$ " $\mathbb{U}$ is λ-closed ". So $\mathbb{P}$ can be seen as the two step iteration of η^+-c.c. forcing and η^+-closed forcing. Because $({}^{<\kappa}\kappa)^L$ is η^+ closed in L, 1 of the above lemma implies that η^+-distributivity of $({}^{<\kappa}\kappa)^L$ is preserved by the first step forcing, and 2 of the above lemma implies that η^+-distributivity of $({}^{<\kappa}\kappa)^L$ is preserved by the second forcing. So $\Vdash_{\mathbb{P}}$ " $({}^{<\kappa}\kappa)^L$ is η^+-distributive ". In general, the above lemma implies that if η is regular and $\mathbb{Q}$ is η-closed in the ground model then two step iteration of η-c.c. forcing and η-closed forcing preserves η-distributivity of $\mathbb{Q}$.

Using Lemma 4.10, it also can be seen that, for each $\lambda < \kappa$, the forcing constructed in Stanley [8] forces that $({}^{<\kappa}\kappa)^L$ is λ-distributive.

On the other hand, the following theorem is shown in Gitik-Magidor-Woodin [7].

Definition 4.11
Let κ be an weakly compact cardinal. Then define the filter WC_κ as follows. We call WC_κ the weakly compact filter over κ.

For each $S \subset \kappa$,
$S \in WC_\kappa \iff \exists R \subseteq V_\kappa\ \forall \langle X, \in, T, Q\rangle$,
if $\langle V_\kappa, \in, S, R\rangle \prec \langle X, \in, T, Q\rangle$ *and* $V_\kappa \subsetneq X$, *then* $\kappa \in T$.
If $\{\eta < \kappa \mid \eta$ *is weakly compact*$\}$ *is a positive set in* WC_κ, *i.e.* $\{\eta < \kappa \mid \eta$ *is not weakly compact*$\}$ *is not in the dual ideal of* WC_κ, *then we call* κ *an weakly compact cardinal of order* 1.

Note: WC_κ is indeed a filter over κ. Moreover it can be easily seen that WC_κ is a normal filter over κ. In general, the weakly compact filter is defined using words of indescribability. (See Kanamori [6].) But it can be easily seen that this definition is equivalent.

Note: If $0^\sharp$ exists then every $i \in I$ is a weakly compact cardinal of order 1 in L. Let $i \in I$ and $S := \{\eta < i \mid \eta$ is weakly compact in $L\}$. Note that i is weakly compact of order 1 in L iff for each $R \subset L_i$, there is $\langle X, \in, T, Q\rangle$ s.t. $\langle L_i, \in, S, R\rangle \prec \langle X, \in, T, Q\rangle$ and $i \in T$. As in the note below the definition of the weakly compactness, we can find i', S' and R' s.t. $i < i' \in I$ and $\langle L_i, \in, S, R\rangle \prec \langle L_{i'}, \in, S', R'\rangle$. Then $L_{i'} \models$ " S' is the set of all weakly compact cardinals $\wedge$ i is weakly compact ". So $i' \in S'$.

Theorem 4.12 (Gitik-Magidor-Woodin [7])
Assume κ *is an weakly compact cardinal of order* 1 *in* L. *Then there is a forcing notion* $\mathbb{P} \in L$ *s.t. if* G *is a* $(L, \mathbb{P})$*-generic filter then in* $L[G]$ *the following hold.*

1. $\kappa = \omega_2$.
2. *There is a closed unbounded* $C \subseteq \kappa$ *s.t. if* $\alpha \in Lim(C)$ *then* α *is inaccessible in* L *and* $C \cap \alpha$ *is faster than every constructible closed unbounded subset of* α.

So by Theorem 3.3,

Theorem 4.13
Assume κ *is an weakly compact cardinal of order* 1 *in* L. *Then there is a forcing notion* $\mathbb{P}$ *s.t.* $\Vdash_{\mathbb{P}}$ "$\kappa = \omega_2\ \wedge\ ({}^{<\kappa}\kappa)^L$ *is not* σ*-distributive* ".

Example 3
Another forcing which adds C of Theorem 3.3 is Radin forcing. Radin forcing adds closed unbounded set C which consists of regular cardinals of the ground model. Moreover if $\alpha \in Lim(C)$ then $C \cap \alpha$ is faster than every closed unbounded subset of α which is in the ground model. Not as in the previous two example, Radin Forcing does not collapse any cardinal. But Radin forcing needs large cardinal axiom which is stronger than the existence of measurable cardinal. In particular, such forcing notion does not exists in L.

References

[1] U. Abraham and S. Shelah, *Forcing closed unbounded sets*, Journal of Symbolic Logic, vol.48 (1983), 643-648.

[2] J. E. Baumgartner, *Iterated forcing*, in: Surveys in Set Theory (A.R.D. Mathias, ed.), London Math. Soc. Lecture Note Series 87 (1983), 1-59.

[3] K. J. Devlin and Ronald B. Jensen, *Marginalia to a Theorem of Silver*, Logic Conference, Kiel, 1974, (G.H. Muller, A.Oberschlep and K. Potthoff , ed.), Springer Lecture Notes, vol.499.

[4] M. Foreman, M. Magidor and S. Shelah $0^\sharp$ *and some forcing principles*, Journal of Symbolic Logic, vol.51 (1986), 39-46.

[5] T. Jech, *Set Theory, Second Edition*, Springer-Verlag Berlin Heidelberg New York, 1997.

[6] A. Kanamori, *The Higher Infinite*, Springer-Verlag New York, Inc., 1994.

[7] M. Gitik, M. Magidor and H. Woodin, *Two weak consequences of* $0^\sharp$., Journal of Symbolic Logic, vol.50 (1985), 597-603.

[8] M. C. Stanley, *Forcing Disabled*, Journal of Symbolic Logic, vol.57 (1992), 1153-1175.