STRUCTURE OF FORMAL SOLUTIONS OF NONLINEAR FIRST ORDER SINGULAR PARTIAL DIFFERENTIAL EQUATIONS IN COMPLEX DOMAIN

MASATAKE MIYAKE AND AKIRA SHIRAI

Abstract. We characterize the convergent or the divergent nature of a given formal solution of nonlinear first order partial differential equations of the form

(SE) $$f(t,x,u,\partial_t u,\partial_x u)=0 \quad \text{with} \quad u(0,x)\equiv 0,$$

where $f(t,x,u,\tau,\xi)$ is holomorphic in a neighborhood of the origin of $\mathbf{C}_t^d \times \mathbf{C}_x^n \times \mathbf{C}_u \times \mathbf{C}_\tau^d \times \mathbf{C}_\xi^n$. We call the equation (SE) is singular in t variables if $f(0,x,0,\tau,0)\equiv 0$ and $f_\xi(0,x,0,\tau,0)\equiv 0$. Under these assumptions, we obtain a criterion for the convergence or the divergence of a formal solution $u(t,x)=\sum_{|\alpha|\geq 1} u_\alpha(x)t^\alpha \in \mathcal{O}_x[[t]]$ whose existence is assumed a priori. Moreover, in the case of divergent solution, we estimate the rate of divergence in term of Gevrey order which is often called the Maillet type theorem.

1. Introduction

We begin with a simple example of nonlinear ordinary differential equations

$$f(t,u,u')\equiv (t-u(t))u'(t)-t^2=0, \quad u(0)=0,$$

where $t\in\mathbf{C}$ denotes the complex variable and $u'(t)=du/dt$. By an easy calculation we see that there are two formal solutions $u(t)=\sum_{n=1}^{\infty} u_n t^n \in \mathbf{C}[[t]]$ such that $u_1=0$ and 1. Then we can prove that the formal solution is convergent if we take $u_1=0$ like a case of regular singular ordinary differential equations, but the formal solution diverges if we take $u_1=1$ as $u_n\sim n!$ like a case of irregular singular ordinary differential equations of the first kind. Thus we understand that the convergent property of formal solutions can not be foreseed from a given equation, since it depends on each formal solution of a singular equation which is defined by $f(0,0,\tau)\equiv 0$ ($\tau\in\mathbf{C}$) in our equation.

In the previous paper [MS], we extended the notion of singular partial differential equations of first order into the case of multi-dimensional t variables, and we characterized the convergence or the divergence of a given formal solution. In this paper, we shall extend the results in [MS] and [S3] by A. Shirai into the equations for which the degeneration occurs for restricted variables which includes many class of linear and nonlinear singular partial differential equations studied by many authors.

We use the following notations in this paper: For $(t, x) = (t_1, \cdots, t_d, x_1, \cdots, x_n) \in \mathbf{C}_t^d \times \mathbf{C}_x^n$ $(d \geq 1, n \geq 0)$, we denote $(\partial_t, \partial_x) = (\partial_{t_1}, \cdots, \partial_{t_d}, \partial_{x_1}, \cdots, \partial_{x_n})$ the symbol of partial differentiations. We denote by $\mathcal{O}_x$ or $\mathbf{C}\{x\}$ the ring of germs of holomorphic functions or the convergent power series in the variable x at $x = 0$. We denote by $\mathcal{O}_x[[t]]$ the ring of formal power series of t with coefficients in $\mathcal{O}_x$. In the case where $n = 0$, we understand $\mathcal{O}_x = \mathbf{C}$, and therefore $\mathcal{O}_x[[t]] = \mathbf{C}[[t]]$. Moreover, we set $\mathcal{M}_x[[t]] = \{u(t, x) \in \mathcal{O}_x[[t]] \,; u(0, x) \equiv 0\}$, that is,

$$u(t, x) \in \mathcal{M}_x[[t]] \Longleftrightarrow u(t, x) = \sum\nolimits_{|\alpha| \geq 1} u_\alpha(x) t^\alpha, \quad u_\alpha(x) \in \mathcal{O}_x. \tag{1.1}$$

We shall study the formal solutions $u(t, x) \in \mathcal{M}_x[[t]]$ of the following nonlinear first order partial differential equation;

$$f(t, x, u, \partial_t u, \partial_x u) = 0 \quad \text{with} \quad u(0, x) \equiv 0. \tag{1.2}$$

Throughout this paper, we assume the following three assumptions:

[**A1**] $f(t, x, u, \tau, \xi)$ $(\tau = (\tau_j) \in \mathbf{C}^d,\ \xi = (\xi_k) \in \mathbf{C}^n)$ is holomorphic in a neighborhood of the origin. Moreover, $f(t, x, u, \tau, \xi)$ is an entire function in τ variables for any fixed t, x, u and ξ in the definite domain.

[**A2**] The equation (1.2) is *singular* in t variables in the sense that

$$f(0, x, 0, \tau, 0) \equiv 0 \ \text{ and } \ \frac{\partial f}{\partial \xi_k}(0, x, 0, \tau, 0) \equiv 0, \quad (k = 1, 2, \ldots, n). \tag{1.3}$$

[**A3**] The equation (1.2) has a formal solution $u(t, x) \in \mathcal{M}_x[[t]]$.

Our purpose in this paper is to characterize the convergence or the divergence of such a formal solution.

In order to state our results we need to prepare some notations.

Let $\varphi_j(x) = \partial_{t_j} u(0, x) \in \mathcal{O}_x$ $(j = 1, \cdots, d)$ and put $\varphi(x) = (\varphi_j(x))$. Then by letting $t = 0$ in the equation (1.2), we get an equation $f(0, x, 0, \varphi(x), 0) \equiv 0$. Since this is a trivial relation from the first assumption in (1.3) of [A2], we can not obtain any information on $\varphi(x)$ from this equation. In order to obtain informations for $\varphi(x)$, we differentiate the equation (1.2) by t_i $(i = 1, 2, \cdots, d)$ and we get the following equations for $\{\varphi_i(x)\}$ from the second assumptions in (1.3) of [A2];

$$\begin{aligned} &\frac{\partial}{\partial t_i} f(t, x, u(t, x), \{\partial_{t_j} u(t, x)\}, \{\partial_{x_k} u(t, x)\})\Big|_{t=0} \\ &\quad \equiv \frac{\partial f}{\partial t_i}(0, x, 0, \varphi(x), 0) + \frac{\partial f}{\partial u}(0, x, 0, \varphi(x), 0)\varphi_i(x) = 0, \end{aligned} \tag{1.4}$$

for $i = 1, 2, \ldots, d$.

We set $\mathbf{a}(x) = (0, x, 0, \varphi(x), 0)$ for the simplicity of notation. Now we define holomorphic functions $a_{ij}(x)$ $(i, j = 1, 2, \ldots, d)$ by

$$a_{ij}(x) = \frac{\partial^2 f}{\partial t_i \partial \tau_j}(\mathbf{a}(x)) + \frac{\partial^2 f}{\partial u \partial \tau_j}(\mathbf{a}(x))\varphi_i(x). \tag{1.5}$$

Then our main result is stated as follows which is a generalization of results in [MS] and [S3] in the case where $n = 0$.

Theorem 1.1. *Under the assumptions* [A1], [A2] *and* [A3], *we have:*

(i) (Convergent Case) *Let $\{\lambda_j\}_{j=1}^d$ be the eigenvalues of the matrix $(a_{ij}(0))_{i,j=1}^d$. Then if $\{\lambda_j\}_{j=1}^d$ satisfies the condition below which we call the Poincaré condition, the formal solution $u(t,x) \in \mathcal{M}_x[[t]]$ is convergent in a neighborhood of the origin:*

$$\mathrm{Ch}(\lambda_1, \ldots, \lambda_d) \not\ni 0 \qquad \text{(Poincaré condition)}, \tag{1.6}$$

where $\mathrm{Ch}(\lambda_1, \ldots, \lambda_d)$ denotes the convex hull of $\{\lambda_1, \ldots, \lambda_d\}$.

(ii) (Divergent Case) *Suppose that $A(x) = (a_{ij}(x))_{i,j=1}^d$ is a nilpotent matrix, and take an integer N with $1 \le N \le d$ such that $A^N(x) \equiv \mathbf{O}$, but $A^j(x) \not\equiv \mathbf{O}$ for $j = 0, \ldots, N-1$, where $\mathbf{O}$ denotes the null matrix. Then if $f_u(\mathbf{a}(0)) \neq 0$, the formal solution $u(t,x) \in \mathcal{M}_x[[t]]$ diverges in general, and it belongs to the Gevrey class of order at most $2N$ in t variables, which means that the formal $2N$-Borel transform of $u(t,x)$, $\sum_{|\alpha|\ge 1} u_\alpha(x)t^\alpha/|\alpha|!^{2N-1}$ is convergent in a neighborhood of the origin.*

The theorem will be proved by reducing the equation (1.2) to an equation which is similar but more general than that studied by Gérard and Tahara [GT] and many others as we shall show below.

We put $v(t,x) = u(t,x) - \sum_{j=1}^d \varphi_j(x)t_j$ $(= O(|t|^2))$. Then by an easy calculation, we can see that $v(t,x)$ satisfies the following nonlinear singular partial differential equation:

$$\left(\sum_{i,j=1}^d a_{ij}(x) t_i \partial_{t_j} + \frac{\partial f}{\partial u}(\mathbf{a}(x))\right) v(t,x) = \sum_{|\alpha|=2} b_\alpha(x) t^\alpha + f_3(t, x, v, \partial_t v, \partial_x v), \tag{1.7}$$

where $b_\alpha(x) \in \mathcal{O}_x$ and $f_3(t, x, v, \tau, \xi)$ is holomorphic in a neighborhood of the origin with Taylor expansion

$$f_3(t, x, v, \tau, \xi) = \sum_{|\alpha|+2p+|q|+2|r|\ge 3} f_{\alpha pqr}(x) t^\alpha v^p \tau^q \xi^r \in \mathcal{O}_x\{t, v, \tau, \xi\}, \tag{1.8}$$

where $\alpha \in \mathbf{N}^d$, $p \in \mathbf{N}$, $q \in \mathbf{N}^d$, $r \in \mathbf{N}^n$ $(\mathbf{N} = \{0, 1, 2, 3, \cdots\})$ and $\mathcal{O}_x\{X\}$ denotes the set of convergent series in all variables x and X.

The theorem will be proved by showing the same statements for $v(t, x)$ which solves the equation (1.7). In the proofs, we first examine the existence of formal solution $v(t, x) = \sum_{|\alpha|\geq 2} v_\alpha(x)t^\alpha = \sum_{L\geq 2} v_L(t, x) \in \mathcal{O}_x[[t]]$, with $v_L(t, x)$ of homogeneous polynomials of degree L in t variables. In the case (i) of the theorem, $\{v_L(t, x)\}$ are uniquely determined except those L's for which the resonance occurs by some α with $|\alpha| = L$ (cf. Remark 1.2 below). Therefore, the existence assumption of a formal solution $v(t, x)$ and the Poincaré condition imply the unique existence of $v_L(t, x)$ for large L, and the essential part of the proof is how we manipulate the Poincaré condition or the nonresonance condition to prove the convergence. It is actually done by showing a majorant estimate of the inverse operator of $P(t, x; \partial_t) = \sum_{i,j=1}^d a_{ij}(x)t_i\partial_{t_j} + f_u(\mathbf{a}(x))$ which appears on the left hand side of (1.7). This enables us to construct a majorant equation (6.23) which is solved by the classical implicit function theorem (cf. Proposition 6.1). In the case (ii) of the theorem, we can easily examine the unique existence of the formal solution $v(t, x) \in \mathcal{O}_x[[t]]$ by the nilpotency condition of $A(x) = (a_{ij}(x))$, but the difficulty lies on the point that the operator $P(t, x; \partial_t)$ is not invertible on $\mathcal{O}_{t,x}$ but is invertible on some space of Gevrey class in t variables (cf. Proposition 7.1 and Remark 7.1). The norm inequality or the majorant relation for the inverse operator P^{-1} established in Proposition 7.1 enables us to construct a majorant partial differential equation (7.13) for which the Gevrey order of solutions is estimated by using a result in [S1] by A. Shirai.

We have to mention that the reduced equation (1.7) is a similar one studied by Gérard and Tahara in their joint works (cf. [GT]). In their works they always assume that the vector field $\sum_{i,j} a_{ij}(x)t_i\partial_{t_j}$ on the left hand side is triangular that $a_{ij}(x) \equiv 0$ if $i > j$, and in the nonlinear term f_3 they assume the existence of variables $\{t_i\partial_{t_j}v\}$ instead of $\partial_t v$ which are not acceptable for a reduced equation from a general equation of singular type. Therefore, we need more careful observation on the invertibility of the vector field and a norm inequality for the inverse operator under which we can employ the majorant method.

Remark 1.1. (About the assumption [A1]) The assumption that the function $f(t, x, u, \tau, \xi)$ is an entire function in τ variable is only for the convenience. Once we fix $\varphi(x) = (\varphi_j(x)) \in {\mathcal{O}_x}^d$ which satisfy the equations (1.4), it is sufficient to assume that f is holomorphic in a neighborhood of $(0, 0, 0, \varphi(0), 0)$.

Remark 1.2. (Nonresonance condition) If $f_u(\mathbf{a}(0))$ satisfies the nonresonance condition, that is,

$$\lambda \cdot \alpha + \frac{\partial f}{\partial u}(\mathbf{a}(0)) \neq 0, \qquad \text{for all } \ |\alpha| \geq 2, \tag{1.9}$$

$(\lambda \cdot \alpha = \sum_{j=1}^d \lambda_j\alpha_j)$, then the theorem does hold for the formal solution $u(t, x) \in \mathbf{C}[[t, x]]$ if we assume the existence of $\varphi(x) = (\varphi_j(x)) \in {\mathcal{O}_x}^d$.

Remark 1.3. (Singular equation) Our definition [A2] or (1.3) on the singular equation corresponds to the one considered by T. Oshima [O] for linear partial differential equations which we will explain in the next section. Especially, our assumption that $f_{\xi_k}(0,x,0,\tau,0) \equiv 0$ $(k = 1, 2, \ldots, n)$ assures that in the reduced equation (1.7) the vector field on the left hand side depends only on ∂_{t_j} $(j = 1, 2 \cdots, d)$. Instead of this assumption, if we assume

$$f_{\xi_k}(0,0,0,\tau,0) \equiv 0 \quad (k = 1, 2 \cdots, n),$$

then we get a sigluar equation of another kind that in the reduced equation the terms $b_k(x)\partial_{x_k}$ with $b_k(0) = 0$ $(k = 1, 2, \cdots, n)$ appear in the vector field.

For such equations, similar problems have been studied in a series of papers [CT], [CL] and [CLT] by Chen, Luo and Tahara where the reduced type equations were studied under more restricted conditions than ours which they called the singular equations of totally characteristic type. The generalization of their results has been studied by A. Shirai. The convergent result has been obtained in [S2] under the generalized Poincaré condition, and the Maillet type theorem has been studied in a preparing paper [S4].

Remark 1.4. (Assumption for $A(x) = (a_{ij}(x))$) In the theorem we assumed that the matrix $A(x)$ is regular in (i) or is nilpotent in (ii), but if we assume that if $A(0)$ singular but $A(x)$ is regular for $x \neq 0$ we meet a different situation for the formal solutions as is seen by the following simple example.

$$-xt\partial_t u + u = f(t,x) \in \mathcal{O}_{t,x}, \quad f(0,x) \equiv 0,$$

where $t, x \in \mathbf{C}$. Let $u(t,x) = \sum_{n=0}^{\infty} u_n(x)t^n$. Then we have

$$(1 - nx)u_n(x) = f_n(x), \quad n \geq 1,$$

which shows the impossibility of the existence of formal solution $u(t,x)$ in $\mathcal{O}_x[[t]]$. In this equation, the unique formal solution $u(t,x) \in \mathbf{C}[[t,x]]$ exists in $\mathcal{O}_t[[x]]$ which is divergent of Gevrey order 2, that is, the role of t and x is converted. For linear equations such considerations are studied in complete form by Hibino [H] which will be explained in the next section in a most easiest form.

2. Related Results in the Linear Equations.

Let us consider the following linear singular partial differential equation with holomorphic coefficients in a neighborhood of the origin $y = 0$ $(y \in \mathbf{C}^m)$

(2.1)
$$\sum_{j=1}^{m} a_j(y)\partial_j u + b(y)u = g(y) \in \mathcal{O}_y, \quad a_j(0) = 0 \ (j = 1, 2, \cdots, m),$$

where $\partial = (\partial_1, \ldots, \partial_m)$ $(\partial_j = \partial/\partial y_j)$.

The essential part of the work by T. Oshima [O] will be stated as follows.

Let $\mathcal{A} := I\{a_1, \ldots, a_m\}$ be the ideal of $\mathcal{O}_y$ generated by the coefficients $\{a_j(y)\}$, and assume there exists d $(1 \le d \le m)$ such that $\mathcal{A} = I\{a_1, \cdots, a_d\}$ and $(\partial_i a_j(0))_{i,j=1}^d$ is invertible. Then by dividing the variables y into two groups by $t_j = y_j$ $(j = 1, \cdots, d)$ and $x_k = y_{d+k}$ $(k = 1, \cdots, m-d)$, we get an equation

$$\sum_{j=1}^{d} a_j(t,x)\partial_{t_j} u + \sum_{k=1}^{m-d} b_k(t,x)\partial_{x_k} u + b(t,x)u = g(t,x), \tag{2.2}$$

where the coefficients satisfy $a_j(0,x) \equiv 0$ with a condition that the matrix $A(x) = (\partial_{t_i} a_j(0,x))_{i,j=1}^d$ is invertible in a neighborhood of the origin $x = 0$, and $b_k(0,x) \equiv 0$.

When we consider the solution $u(t,x) \in \mathcal{M}_x[[t]]$ as in the nonlinear equations, the assumption [A2] is satisfied if and only if $g(0,x) \equiv 0$ which is a trivial restriction followed from the equation. Therefore our result is applicable in this case.

Recently, M. Hibino [H] studied the singular equation (2.1) without any other assumption, but under the assumption of the nonresonance condition between $b(0)$ and the nonzero eigenvalues of the Jacobi matrix

$$D_y a(0) = \frac{\partial(a_1, \cdots, a_m)}{\partial(y_1, \cdots, y_m)}(0),$$

which assures the unique existence of formal solution in $\mathbf{C}[[y]]$. Therefore when the Jacobi matrix $D_y a(0)$ is nilpotent, he only assumed that $b(0) \neq 0$. Now one of his results is stated as follows.

Theorem. ([H]) *Under the above mentioned conditions, let N be the maximal dimension of generalized eigenspaces associated with zero eigenvalues of the Jacobi matrix $D_y a(0)$ and or $N = 1/2$ if there does not exist zero eigenvalues. Then, the formal solution $u(y) \in \mathbf{C}[[y]]$ belongs to a Gevrey space $\mathcal{G}^{2N}$, which means $\sum_{\alpha \in \mathbf{N}^m} u_\alpha \, y^\alpha / |\alpha|!^{2N-1} \in \mathcal{O}_y$.*

Therefore, Theorem 1.1 is an extension of a part of Hibino's results to nonlinear equations.

3. Example.

Example 3.1. Let $(t,x) \in \mathbf{C}^2$. We consider the following equation:

$$\begin{cases} \{u - a(x)t\}u_t - uu_x = p(x)t^2, \\ u(0,x) \equiv 0, \end{cases} \tag{3.1}$$

where $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$ and $a(x)$, $p(x) \in \mathcal{O}_x$ with $a(0) \neq 0$. Let $u(t,x) = \sum_{n=1}^{\infty} u_n(x) t^n$ be a formal solution. Then $u_1(x)$ should satisfy

$$\{u_1(x) - a(x)\}u_1(x) \equiv 0.$$

Therefore, $u_1(x) \equiv 0$ or $u_1(x) = a(x)$, and after a choice of $u_1(x)$ we can see that the formal solution is determined uniquely.

The case where $u_1(x) \equiv 0$. The formal solution $u(t,x)$ satisfies

$$a(x)tu_t = -p(x)t^2 + uu_t - uu_x, \qquad u = O(t^2).$$

This equation is a special case of the equation (1.7), and $u(t,x)$ is convergent by Theorem 1.1, (i), since $a(0) \neq 0$ which corresponds to the Poincaré condition.

The case where $u_1(x) = a(x)$. Let $u(t,x) = a(x)t + v(t,x)$ $(v = O(t^2))$. Then $v(t,x)$ satisfies $v(a(x)+v_t) - (a(x)t+v)(a'(x)t+v_x) = p(x)t^2$, that is,

$$a(x)v = (p(x) + a(x)a'(x))t^2 - vv_t + a'(x)tv + a(x)tv_x + vv_x.$$

This equation is a special case of the equation (1.7) with $a_{ij}(x) \equiv 0$ which can be applied Theorem 1.1, (ii). Therefore, when $p(x)+a(x)a'(x) \not\equiv 0$, the uniquely determined formal solution $v(t,x)$ belongs to a class of Gevrey order 2 in t variable. On the other hand, if $p(x) + a(x)a'(x) \equiv 0$, we have $v(t,x) \equiv 0$.

4. Preparations to Prove Theorem 1.1.

In this section, we shall prepare some notations, definitions and lemmas, which will be used in the proof of Theorem 1.1.

- $D_{z_0}(R) = \{x = (x_1, \dots, x_n) \in \mathbf{C}^n \ ; \ |x_j - z_0| \leq R, \ j = 1, 2, \dots, d, \ z_0 \in \mathbf{C}\}$.
- $\mathcal{O}_{z_0}(R)$: the set of holomorphic functions on $x \in D_{z_0}(R)$.
- $\mathbf{C}[t]_L = \{u_L(t) = \sum_{|\alpha|=L} u_\alpha t^\alpha \ ; \ u_\alpha \in \mathbf{C}\}$. (Homogeneous polynomials of order L)
- $\mathcal{O}_{z_0}(R)[t]_L = \{u_L(t,x) = \sum_{|\alpha|=L} u_\alpha(x) t^\alpha \ ; \ u_\alpha(x) \in \mathcal{O}_{z_0}(R)\}$.

Definition 4.1 (**s**-Borel transform and Gevrey space $\mathcal{G}^{\mathbf{s}}$)**.** Let $\mathbf{R}_{\geq 1} = \{x \in \mathbf{R} \ ; \ x \geq 1\}$. For d dimensional real vector $\mathbf{s} = (s_1, s_2, \dots, s_d) \in (\mathbf{R}_{\geq 1})^d$ and a formal power series $f(t,x) = \sum_{\alpha \in \mathbf{N}^d} f_\alpha(x) t^\alpha \in \mathcal{O}_x[[t]]$, we define the **s**-Borel transform $\mathcal{B}^{\mathbf{s}}(f)(t,x)$ of $f(t,x)$ by

$$\mathcal{B}^{\mathbf{s}}(f)(t,x) := \sum_{\alpha \in \mathbf{N}^d} f_\alpha(x) \frac{|\alpha|!}{(\mathbf{s}\cdot\alpha)!} t^\alpha, \tag{4.1}$$

where $\mathbf{s}\cdot\alpha = \sum_{j=1}^d s_j \alpha_j$ and $(\mathbf{s}\cdot\alpha)! = \Gamma(\mathbf{s}\cdot\alpha + 1)$ by the Gamma function.

We say that $f(t,x) \in \mathcal{G}^{\mathbf{s}}$ if $\mathcal{B}^{\mathbf{s}}(f)(t,x) \in \mathbf{C}\{t,x\}$, and $\mathbf{s}$ is called the Gevrey order in t variables.

We introduce the **s**-norm of $u_L(t) = \sum_{|\alpha|=L} u_\alpha t^\alpha \in \mathbf{C}[t]_L$ by

$$\begin{aligned} ||u_L||_{\mathbf{s}} &:= \inf\{C > 0 \ ; \ \mathcal{B}^{\mathbf{s}}(u_L)(t) \ll C(t_1 + \cdots + t_d)^L\} \\ &= \max_{|\alpha|=L} \left\{ |u_\alpha| \frac{\alpha!}{(\mathbf{s}\cdot\alpha)!} \right\}, \qquad (\alpha! = \alpha_1! \cdots \alpha_d!). \end{aligned} \tag{4.2}$$

Lemma 4.1. *Let $f(t,x)=\sum_{\alpha\in\mathbf{N}^d} f_\alpha(x)t^\alpha \in \mathcal{O}_0(R)[[t]]$ and assume $\mathbf{s}=(s,\cdots,s)$ $\in(\mathbf{R}_{\geq 1})^d$. For a regular matrix $Q(x)=(Q_{ij}(x))\in GL(d,\mathcal{O}_0(R))$, the function $g(\tau,x):=f(\tau Q(x),x)$ belongs to $\mathcal{G}^{\mathbf{s}}$ in τ variables if and only if $f(t,x)$ belongs to $\mathcal{G}^{\mathbf{s}}$ in t variables.*

Proof. We only prove the sufficient condition, since the necessary condition follows from it. By the definition, $f(t,x)=\sum f_\alpha(x)t^\alpha\in\mathcal{G}^{\mathbf{s}}$ in t variables if and only if there exist positive constants A and B such that

$$\max_{x\in D_0(R)}|f_\alpha(x)|\leq AB^{|\alpha|}\frac{(s|\alpha|)!}{|\alpha|!},\qquad \text{for all}\ \ \alpha\in\mathbf{N}^d. \tag{4.3}$$

We recall $g(\tau,x)=\sum f_\alpha(x)(\tau Q(x))^\alpha$. For any fixed $x\in D_0(R)$, $(\tau Q(x))^\alpha$ is estimated by

$$\begin{aligned}(\tau Q(x))^\alpha &= \prod_{j=1}^d\left(\sum_{k=1}^d \tau_k Q_{kj}(x)\right)^{\alpha_j}\\ &\ll \left(\max_{i,j}\max_{x\in D_0(R)}\{|Q_{ij}(x)|\}\right)^{|\alpha|}(\tau_1+\cdots+\tau_d)^{|\alpha|}.\end{aligned}$$

We set $C=\max_{i,j}\max_{x\in D_0(R)}|Q_{ij}(x)|$. Then by (4.3) we have

$$f_\alpha(x)(\tau Q(x))^\alpha\ll A(BC)^{|\alpha|}\,\frac{(s|\alpha|)!}{|\alpha|!}\,(\tau_1+\cdots+\tau_d)^{|\alpha|},\quad \text{for all } x\in D_0(R).$$

This implies

$$\begin{aligned}\mathcal{B}^{\mathbf{s}}\{f_\alpha(x)(\tau Q(x))^\alpha\} &\ll A(BC)^{|\alpha|}\frac{(s|\alpha|)!}{|\alpha|!}\sum_{|\beta|=|\alpha|}\frac{|\alpha|!}{\beta!}\frac{|\alpha|!}{(s|\alpha|)!}\tau^\beta\\ &= A(BC)^{|\alpha|}\,(\tau_1+\cdots+\tau_d)^{|\alpha|},\quad \text{for all } x\in D_0(R),\end{aligned}$$

Therefore, for all $x\in D_0(R)$ we have

$$\mathcal{B}^{\mathbf{s}}(g)(\tau,x)\ll A\sum_{\alpha\in\mathbf{N}^d}(BC)^{|\alpha|}(\tau_1+\cdots+\tau_d)^{|\alpha|}\in\mathcal{O}_\tau.$$

This proves that $g(\tau,x)\in\mathcal{G}^{\mathbf{s}}$ in τ variables. □

5. Proof of Theorem 1.1, (i).

We put $v(t,x)=u(t,x)-\sum_{j=1}^d\varphi_j(x)t_j\in\mathcal{M}_x[[t]]$ which satisfies $v(t,x)=O(|t|^2)$. Then, as stated in Introduction, it is easily examined that $v(t,x)$ satisfies the following singular equation:

$$\left(\sum_{i,j=1}^d a_{ij}(x)t_i\partial_{t_j}+c(x)\right)v(t,x)=\sum_{|\alpha|=2}b_\alpha(x)t^\alpha+f_3(t,x,v,\partial_t v,\partial_x v), \tag{5.1}$$

with $a_{ij}(x), c(x), b_\alpha(x) \in \mathcal{O}_x$. Here we remark that $(a_{ij}(0))_{i,j=1}^d$ is a regular matrix with eigenvalues $\{\lambda_j\}_{j=1}^d$ which satisfy the Poincaré condition (1.6), $c(x) = f_u(\mathbf{a}(x))$ and $f_3(t, x, v, \tau, \xi)$ is holomorphic in a neighborhood of the origin with Taylor expansion

$$f_3(t, x, v, \tau, \xi) = \sum_{|\alpha|+2p+|q|+2|r|\geq 3} f_{\alpha pqr}(x) t^\alpha v^p \tau^q \xi^r, \tag{5.2}$$

where $\alpha \in \mathbf{N}^d$, $p \in \mathbf{N}$, $q \in \mathbf{N}^d$ and $r \in \mathbf{N}^n$.

By the Poincaré condition (1.6), there exists a positive integer $K \geq 2$ such that

$$\left| \sum_{j=1}^d \lambda_j \alpha_j + c(0) \right| \geq C_0 |\alpha|, \quad |\alpha| \geq K \tag{5.3}$$

holds by some positive constant $C_0 > 0$. We take and fix such K.

Once again we set $w(t, x) = v(t, x) - \sum_{|\alpha|=2}^{K-1} u_\alpha(x) t^\alpha$ $(= O(|t|^K))$ as a new unknown function. Then $w(t, x)$ satisfies a singular equation of the following form:

$$\left(\sum_{i,j=1}^d a_{ij}(x) t_i \partial_{t_j} + c(x) \right) w = \sum_{|\alpha|=K} d_\alpha(x) t^\alpha + f_{K+1}(t, x, w, \partial_t w, \partial_x w), \tag{5.4}$$

where $d_\alpha(x) \in \mathcal{O}_x$ and $f_{K+1}(t, x, u, \tau, \xi)$ is holomorphic in a neighborhood of the origin with Taylor expansion

$$f_{K+1}(t, x, u, \tau, \xi) = \sum_{|\alpha|+Kp+(K-1)|q|+K|r|\geq K+1} f_{\alpha pqr}(x) t^\alpha u^p \tau^q \xi^r. \tag{5.5}$$

Therefore, the proof of Theorem 1.1, (i), is reduced to prove the following Theorem:

Theorem 5.1. *Under the condition* (5.3)*, the equation* (5.4) *with* $w(t, x) = O(|t|^K)$ *has a unique formal solution which converges in a neighborhood of the origin.*

We shall give two proofs of this Theorem, since both are seemed interesting. The first one will be given in the next section which is based on the view point as an evolution of (5.4) in t variables. The second one will be given in Appendix (Section 8) where all the variable t and x are considered to play the same role in a sense.

6. Proof of Theorem 5.1

By a linear change of t variables which brings $(a_{ij}(0))$ to its Jordan canonical form, the equation (5.4) is reduced to the following one:

$$(\Lambda + \Delta + A) w(t, x) = \sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha + g_{K+1}(t, x, w, \partial_t w, \partial_x w), \tag{6.1}$$

with $w(t,x) = O(|t|^K)$, where

$$\Lambda = \sum_{j=1}^{d} \lambda_j t_j \partial_{t_j} + c(0), \qquad \Delta = \sum_{j=1}^{d-1} \delta_j t_{j+1} \partial_{t_j}, \tag{6.2}$$

$$A \equiv A(x) = \sum_{i,j=1}^{d} \alpha_{ij}(x) t_i \partial_{t_j} + b(x), \quad (\alpha_{ij}(0) = 0, b(0) = 0),$$

and g_{K+1} is holomorphic in a neighborhood of the origin with the same Taylor expansion with f_{K+1}.

Remark 6.1. In the part Δ, it is normally considered that $\delta_j = 0$ or 1. However, we can take $\{\delta_j\}$ are as small as we want. Indeed, if we take a change of variables by $\widehat{t}_j = \varepsilon^j t_j$, then δ_j is replaced by $\varepsilon \delta_j$.

For the proof our theorem, the following proposition plays an essential role to employ the majorant method.

Proposition 6.1. *Let us consider the linear operator $P = \Lambda + \Delta + A$.*

(i) *For all $L \geq K$, the mapping $P : \mathcal{O}_0(R)[t]_L \longrightarrow \mathcal{O}_0(R)[t]_L$ is invertible for sufficiently small $R > 0$.*

(ii) *For $u(t,x) \in \mathcal{O}_0(R)[t]_L$, if a majorant relation $u(t,x) \ll W(x)(t_1 + \cdots + t_d)^L$ does hold by a function $W(x)$ with non negative Taylor coefficients, then there exists a positive constant $F > 0$ independent of L such that*

$$\begin{aligned} P^{-1}u(t,x) &\ll \frac{1}{L}\frac{F}{R-X} W(x)(t_1 + \cdots + t_d)^L \\ &= (T\partial_T)^{-1} \frac{F}{R-X} W(x)(t_1 + \cdots + t_d)^L, \end{aligned} \tag{6.3}$$

where $T = t_1, \ldots + t_d$ and $X = x_1 + \cdots + x_n$.

Remark 6.2. By the above lemma, a majorant operator of P^{-1} on the space of homogeneous polynomials in t variables with holomorphic coefficients in x variables is given by

$$P^{-1} \ll (T\partial_T)^{-1} \frac{F}{R-X} \ll \frac{F}{R-X}. \tag{6.4}$$

The precise definition of majorant operator will be given later.

Proof of Proposition 6.1, (i). First we prove the invertibility of the operator $P = \Lambda + \Delta + A$ on $\mathbf{C}[t]_L$ $(L \geq K)$ for any fixed $x \in D_0(R)$ which is considered as a parameter.

Let us, first, consider a linear mapping $T = (T_{\alpha\beta})_{|\alpha|=|\beta|=L} : \mathbf{C}[t]_L \longrightarrow \mathbf{C}[t]_L$ defined by

$$\sum_{|\beta|=L} u_\beta t^\beta \mapsto \sum_{|\alpha|=L} \left\{ \sum_\beta T_{\alpha\beta} u_\beta \right\} t^\alpha.$$

We take the norm $\|\cdot\|_{\mathbf{1}}$ on $\mathbf{C}[t]_L$ with $\mathbf{1} = (1, \cdots, 1)$ defined by (4.2), that is, for $u(t) = \sum_{|\beta|=L} u_\beta t^\beta \in \mathbf{C}[t]_L$, $\|u\|_{\mathbf{1}} = \inf\{C > 0\,; u(t) \ll C(t_1 + \cdots t_d)^L\} = \sup_\beta\{|u_\beta|\beta!/|\beta|!\}$.

Then the operator norm $\|T\|_{\mathbf{1}}$ of T is given by

$$||T||_{\mathbf{1}} = \max_{|\alpha|=L} \sum_{|\beta|=L} |T_{\alpha\beta}| \frac{\alpha!}{\beta!}. \tag{6.5}$$

We omit the proof, since it is elementary.

We return to the proof of the lemma. We consider the mapping $\Lambda = \sum_{j=1}^d \lambda_j \partial_{t_j} + c(0) : \mathbf{C}[t]_L \to \mathbf{C}[t]_L$ $(L \geq K)$. Since the matrix representation T of Λ is a diagonal matrix with $T_{\alpha\alpha} = \sum_{j=1}^d \lambda_j \alpha_j + c(0) \neq 0$ $(L \geq K)$ which implies the invertibility of Λ, and by the condition (5.3) we get the following operator norm of Λ^{-1}

$$||\Lambda^{-1}||_{\mathbf{1}} \leq (C_0 L)^{-1}, \tag{6.6}$$

where $C_0 > 0$ is the constant in (5.3).

Next, we consider the estimate of the operator norms of Δ and A on $\mathbf{C}[t]_L$ for any fixed $x \in D_0(R)$. By computing the norms of $\Delta u(t)$ and $Au(t)$ for $u(t) \in \mathbf{C}[t]_L$, their operator norms are estimated as follows:

$$||\Delta||_{\mathbf{1}} \leq L \times \max_{j=1,2,\dots,d-1} |\delta_j|, \tag{6.7}$$

and

$$||A||_{\mathbf{1}} \leq L \times \max_i \sum_{j=1}^d \sup_{x \in D_0(R_0)} |\alpha_{ij}(x)| + \sup_{x \in D_0(R_0)} |b(x)|, \tag{6.8}$$

By Remark 6.1, we may assume the constants δ_j in Δ are arbitrary small as we want, and we recall that $\alpha_{ij}(x)$ and $b(x)$ vanish at $x = 0$. Therefore, by taking $R > 0$ sufficiently small we may assume that

$$\|\Delta + A\|_1 < C_0 L, \quad \text{for all } x \in D_0(R). \tag{6.9}$$

Therefore, for all $x \in D_0(R)$ we have $||\Lambda^{-1}(\Delta + A)||_{\mathbf{1}} < 1$ on $\mathbf{C}[t]_L$ $(L \geq K)$ which proves the invertibility of $P = \Lambda + \Delta + A$ on $\mathbf{C}[t]_L$ $(L \geq K)$.

Now the holomorphic dependency of P^{-1} on $x \in D_0(R)$ follows immediately. In fact, we first notice that P^{-1} is given by the Neumann series

$$P^{-1} = (\Lambda + \Delta + A)^{-1} = \left\{\sum_{n=0}^{\infty} (-\Lambda(\Delta + A)^n\right\} \Lambda^{-1}.$$

Here we note that the matrix representation of the operator $\Lambda^{-1}(\Delta + A)$ $(A = A(x))$ depends holomorphically on $x \in D_0(R)$ and is constructive uniformly on the same domain. Therefore the convergence of the Neumann series is uniform on $D_0(R)$, which completes the proof. $\square$

For the proof of the majorant relation (6.3) in Proposition 6.1, (ii), we introduce a notion of majorant operator.

Definition 6.1 (Majorant Operator)**.** Let $a(x) = (a_{\alpha\beta}(x))_{|\alpha|=|\beta|=L}$ and $A(x) = (A_{\alpha\beta}(x))_{|\alpha|=|\beta|=L}$ be linear operators on $\mathcal{O}_0(R)[t]_L$. We say that $A(x)$ is a majorant operator of $a(x)$, if $a_{\alpha\beta}(x) \ll A_{\alpha\beta}(x)$ hold for all $|\alpha| = |\beta| = L$. If $A(x)$ is a majorant operator of $a(x)$, we write this relation by $a(x) \ll A(x)$.

Remark 6.3. The following relations are obviously hold:

- If $a(x) \ll A(x)$ and $b(x) \ll B(x)$, then $a(x)b(x) \ll A(x)B(x)$.
- Let $a(x) \ll A(x)$. If two formal power series $u(t,x)$ and $U(t,x)$ satisfy $u(t,x) \ll U(t,x)$, then $a(x)u(t,x) \ll A(x)U(t,x)$.

Proof of Proposition 6.1, (ii) We set $\Lambda_0 = \sum_{j=1}^{d} t_j \partial_{t_j}$. By the condition (5.3) and the above definition, we have $\Lambda^{-1} \ll (C_0\Lambda_0)^{-1}$. For a formal power series $f(x) = \sum f_\alpha x^\alpha \in \mathbf{C}[[x]]$, we define $|f|(x)$ by $|f|(x) := \sum |f_\alpha| x^\alpha$. By using this notation, we have a majorant relation

$$|\Delta| + |A| \gg \Delta + A,$$

where $|\Delta| := \sum_{j=1}^{d-1} |\delta_j| t_{j+1}\partial_{t_j}$ and $|A| := \sum_{i,j=1}^{d} |\alpha_{ij}|(x) t_i \partial_{t_j} + |b|(x)$.

Now we take $R'(\le R)$ such that

$$||(C_0\Lambda_0)^{-1}(|\Delta| + |A|)||_{\mathbf{1}} < 1, \quad \text{for all } x \in D_0(R'). \tag{6.10}$$

By this inequality, a majorant operator of P^{-1} is given by

$$\begin{aligned} P^{-1} &= (\Lambda + \Delta + A)^{-1} = \left\{ \sum_{n=0}^{\infty} (-\Lambda^{-1}(\Delta + A))^n \right\} \Lambda^{-1} \\ &\ll \left\{ \sum_{n=0}^{\infty} ((C_0\Lambda_0)^{-1}(|\Delta| + |A|))^n \right\} (C_0\Lambda_0)^{-1} \\ &= \{I - (C_0\Lambda_0)^{-1}(|\Delta| + |A|)\}^{-1}(C_0\Lambda_0)^{-1} \\ &= \{(C_0\Lambda_0) - (|\Delta| + |A|)\}^{-1} =: Q^{-1}, \end{aligned}$$

where I denotes the identity operator.

Let $a(x) = (a_{\alpha\beta}(x))_{|\alpha|=|\beta|=L}$ and $C(x) = (C_{\alpha\beta}(x))_{|\alpha|=|\beta|=L}$ ($C_{\alpha\beta}(x) \gg 0$) be the matrix representations of the linear operators P^{-1} and $\{I - (C_0\Lambda_0)^{-1}(|\Delta| + |A|)\}^{-1}$, respectively. Then the matrix representation $A(x) = (A_{\alpha\beta}(x))_{|\alpha|=|\beta|=L}$ of the linear operator Q^{-1} is given by $(A_{\alpha\beta}(x)) = (C_0 L)^{-1} \times (C_{\alpha\beta}(x))$. Therefore, we have the majorant relation $a_{\alpha\beta}(x) \ll (C_0L)^{-1} C_{\alpha\beta}(x)$ for all $|\alpha| = |\beta| = L$.

Now we assume a majorant relation

$$u(t,x) = \sum_{|\alpha|=L} u_\alpha(x) t^\alpha \ll W(x)(t_1 + \cdots + t_d)^L, \quad (W(x) \gg 0),$$

which means $u_\alpha(x) \ll W(x) L!/\alpha!$ for all $|\alpha| = L$.

We put $P^{-1}u(t,x) = \sum_{|\alpha|=L} v_\alpha(x)t^\alpha$. Since the matrix representation of P^{-1} is $a(x) = (a_{\alpha\beta}(x))_{|\alpha|=|\beta|=L}$, we have

$$v_\alpha(x) = \sum_{|\beta|=L} a_{\alpha\beta}(x)u_\beta(x) \ll \frac{W(x)}{C_0 L} \sum_{|\beta|=L} C_{\alpha\beta}(x)\frac{L!}{\beta!},$$

that is,

$$v_\alpha(x)\frac{\alpha!}{L!} \ll \frac{W(x)}{C_0 L} \sum_{|\beta|=L} C_{\alpha\beta}(x)\frac{\alpha!}{\beta!}. \tag{6.11}$$

In order to prove (6.3), it is sufficient to prove the existence of a positive constant F and a constant $R''(\le R')$ independent of L such that

$$\sum_{|\beta|=L} C_{\alpha\beta}(x)\frac{\alpha!}{\beta!} \ll \frac{F}{R''-X}, \quad |\alpha| = L, \tag{6.12}$$

where $X = x_1 + \cdots + x_n$. In fact, if once we prove (6.12), then we have

$$\begin{aligned} P^{-1}u(t,x) &= \sum_{|\alpha|=L} v_\alpha(x)t^\alpha = \sum_{|\alpha|=L} v_\alpha(x)\frac{\alpha!}{L!}\frac{L!}{\alpha!}t^\alpha \\ &\ll \sum_{|\alpha|=L} \frac{W(x)}{C_0 L}\left(\sum_{|\beta|=L} C_{\alpha\beta}(x)\frac{\alpha!}{\beta!}\right)\frac{L!}{\alpha!}t^\alpha \\ &\ll \frac{W(x)}{C_0 L}\frac{F}{R-X}(t_1 + \cdots + t_d)^L, \end{aligned}$$

which we want to prove. Thus the proof is reduced to establish (6.12).

Proof of (6.12). It is sufficient to prove the existence of $R''(\le R')$ such that the following estimate does hold by a positive constant $C > 0$ independent of L.

$$\max_\alpha \max_{x\in D_0(R'')} \sum_{|\beta|=L} |C_{\alpha\beta}(x)|\frac{\alpha!}{\beta!} \le C. \tag{6.13}$$

In fact, by (6.13), $f_\alpha(x) = \sum_\beta C_{\alpha\beta}(x)\alpha!/\beta!$ is holomorphic on $D_0(R'')$ and $|f_\alpha(x)| \le C$. Then by Cauchy's integral formula we can prove the following majorant relations.

$$f_\alpha(x) \ll \frac{C(R'')^n}{(R''-x_1)(R''-x_2)\cdots(R''-x_n)} \ll \frac{CR''}{R''-X} \tag{6.14}$$

by $X = x_1 + \cdots + x_n$.

Now let us prove (6.13). We use the following notations.

$$||x|| = (|x_1|, \dots, |x_n|) \in (\mathbf{R}_{\ge 0})^n, \quad ||R|| = (R, \dots, R) \in (\mathbf{R}_{\ge 0})^n.$$

Since $C_{\alpha\beta}(x) \gg 0$, we have

$$\max_{x\in D_0(R)} |C_{\alpha\beta}(x)| \le \max_{x\in D_0(R)} C_{\alpha\beta}(||x||) \le C_{\alpha\beta}(||R||).$$

Now we take a positive constant $R'' > 0$ such that

$$||(C_0\Lambda_0)^{-1}(|\Delta| + |A|)|_{x=||R''||}||_{\mathbf{1}} < 1,$$

where $\|R''\| = (R'', \cdots, R'') \in (\mathbf{R}_{>0})^n$ and

$$(|\Delta| + |A|)|_{x=\|R''\|} \equiv |\Delta| + \sum_{i,j} |\alpha_{ij}|(\|R''\|)t_j\partial_{t_j} + |b|(\|R''\|) : \mathbf{C}[t]_L \to \mathbf{C}[t]_L,$$

the restriction mapping at $x = \|R''\|$.

Then for all $|\alpha| = L$ we have

$$\begin{aligned}
\sum_{|\beta|=L} \max_{x\in D_0(R'')} |C_{\alpha\beta}(x)|\frac{\alpha!}{\beta!} &\leq \sum_{|\beta|=L} C_{\alpha\beta}(||R''||)\frac{\alpha!}{\beta!} \\
&\leq \max_{|\alpha|=L} \sum_{|\beta|=L} C_{\alpha\beta}(||R''||)\frac{\alpha!}{\beta!} \\
&= \left\|\left\{I - (C_0\Lambda_0)^{-1}(|\Delta| + |A|)|_{x=||R''||}\right\}^{-1}\right\|_{\mathbf{1}} \\
&\leq \frac{1}{1 - ||(C_0\Lambda_0)^{-1}(|\Delta| + |A|)|_{x=||R''||}||_{\mathbf{1}}} < \infty,
\end{aligned}$$

where we used the norm equality (6.5) at the place of the equality. $\square$

Proof of Theorem 5.1. We take a small positive constant $R > 0$ such that the functions in the equation are holomorphic on $D_0(R)$ and that Proposition 6.1 does hold. By this choice of R we easily see that the formal solution $w(t,x) \in \mathcal{M}_x[[t]]$ with $w(t,x) = O(|t|^K)$ of the equation (6.1) exists uniquely by the invertibility of P on every $\mathcal{O}_0(R)[t]_L$ $(L \geq K)$. Indeed, the formal solution $w(t,x) = \sum_{L\geq K} w_L(t,x)$ $(w_L(t,x) \in \mathcal{O}_0(R)[t]_L)$ are determined inductively on L. Therefore, we have only to prove the convergence of this formal solution $w(t,x)$.

Let $U(t,x) = Pw(t,x)$ be a new unknown function. Then $U(t,x)$ satisfies the following equation by (6.1):

(6.15)
$$\begin{cases} U(t,x) = \sum_{|\alpha|=K} \zeta_\alpha(x)t^\alpha + g_{K+1}(t,x,P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U), \\ U(t,x) = O(|t|^K). \end{cases}$$

In order to prove the convergence of formal solution $U(t,x)$, we prepare majorant functions (which are convergent) as follows.

$$\sum_{|\alpha|=K} \zeta_\alpha(x)t^\alpha \ll \frac{A}{(R-X)^K}T^K, \quad (T = t_1 + \cdots + t_d,\ X = x_1 + \cdots + x_n),$$

$$
\begin{aligned}
&g_{K+1}(t,x,u,\tau,\xi)\\
&\quad\ll \sum_{|\alpha+Kp+(K-1)|q|+K|r|\geq K+1} \frac{G_{\alpha pqr}}{(R-X)^{|\alpha|+p+|q|+|r|}} T^{|\alpha|}u^p\tau^q\xi^r\\
&\quad=: G_{K+1}(T,X,u,\tau,\xi).
\end{aligned}
$$

We recall the majorant relations (6.3) in Proposition 6.1 and (6.4), and notice an elementary majorant relation of operators that $\partial_{t_j}(T\partial_T)^{-1} \ll 1/T$. We consider the following equation:

(6.16)

$$
\begin{cases}
W(T,X) = \dfrac{A}{(R-X)^K}T^K + G_{K+1}\Bigg(T, X, \dfrac{F}{R-X}W, \\
\qquad\qquad \left\{\dfrac{F}{R-X}\dfrac{W}{T}\right\}_{j=1}^{d}, \left\{\partial_{x_k}(T\partial_T)^{-1}\dfrac{F}{R-X}W\right\}_{k=1}^{n}\Bigg), \\
W(T,X) = O(T^K),
\end{cases}
$$

where F is the same positive constant in (6.3). We note that $\partial_{x_k} = \partial_X$ in the above equation.

By this construction of the equation, we easily see that the formal solution $W(T,X) \in \mathcal{O}_X[[T]]$ (which is uniquely determined) is a majorant function of $U(t,x)$, that is, $W(t_1+\cdots+t_d, x_1+\cdots+x_n) \gg U(t,x)$ holds. Therefore, it is sufficient to prove the convergence of $W(T,X)$. We put $W(T,X) = \sum_{L\geq K} W_L(X)T^L$ and by substituting this into (6.16), we obtain the following recursion formulas:

$$
W_K(X) = \frac{A}{(R-X)^K}, \tag{6.17}
$$

and for $L \geq K+1$,

$$
\begin{aligned}
W_L(X) \;=\; &\sum_{V(\alpha,p,q,r)\geq K+1} \frac{G_{\alpha pqr}}{(R-X)^{|\alpha|+p+|q|+|r|}} {\sum}' \prod_{l=1}^{p} \frac{F}{R-X} W_{L_l}(X)\\
&\times \prod_{j=1}^{d}\prod_{l=1}^{q_j} \frac{F}{R-X} W_{M_{jl}}(X) \prod_{k=1}^{n}\prod_{l=1}^{r_k} \frac{1}{N_{kl}} \partial_{x_k} \frac{F}{R-X} W_{N_{kl}}(X),
\end{aligned} \tag{6.18}
$$

where

$$
V(\alpha,p,q,r) = |\alpha| + Kp + (K-1)|q| + K|r|, \tag{6.19}
$$

and summation $\sum'$ is taken over

$$
|\alpha| + \sum_{l=1}^{p} L_l + \sum_{j=1}^{d}\sum_{l=1}^{q_j}(M_{jl}-1) + \sum_{k=1}^{n}\sum_{l=1}^{r_k} N_{kl} = L. \tag{6.20}
$$

By these recursion formulas, we can prove the following lemma:

Lemma 6.1. *The coefficients* $\{W_L(X)\}_{L\geq K}$ *are given by*

$$W_L(X)=\sum_{j=K}^{10L-9K}\frac{W_{Lj}}{(R-X)^j},\quad \text{by some}\quad W_{Lj}\geq 0. \tag{6.21}$$

Proof of Lemma 6.1. The expression (6.21) can be proved inductively on L by estimating the powers of $1/(R-X)$. The case $L=K$ is trivial, and consider the case $L>K$. The lower bound is easy to prove, so we omit it. The upper bound is estimated by

$$\begin{aligned}
j &\leq |\alpha|+p+|q|+|r|+\sum_{l=1}^{p}(10L_l-9K+1)\\
&\quad +\sum_{j=1}^{d}\sum_{l=1}^{q_j}(10M_{jl}-9K+1)+\sum_{k=1}^{n}\sum_{l=1}^{r_k}(10N_{kl}-9K+2)\\
&= 10L-9\{|\alpha|+Kp+(K-1)|q|+K|r|\}+2p+3|q|+3|r|\\
&\leq 10L-9\{|\alpha|+Kp+(K-1)|q|+K|r|\}\\
&\quad +\frac{3}{K-1}\{|\alpha|+Kp+(K-1)|q|+K|r|\}\\
&= 10L-\left(9-\frac{3}{K-1}\right)\{|\alpha|+Kp+(K-1)|q|+K|r|\}\\
&\leq 10L-\left(9-\frac{3}{K-1}\right)(K+1)\\
&= 10L-9K+\left(\frac{3(K+1)}{K-1}-9\right)\\
&\leq 10L-9K.
\end{aligned}$$

This proves the lemma. $\square$

By the representation (6.21), we have the following majorant relation:

$$\begin{aligned}
\partial_{x_k}(T\partial_T)^{-1}\frac{F}{R-X}W(T,X) &= \sum_{L\geq K}\sum_{j=K}^{10L-9K}\frac{j+1}{L}\frac{FW_{Lj}}{(R-X)^{j+2}}T^L\\
&\ll \frac{10F}{(R-X)^2}W(T,X).
\end{aligned} \tag{6.22}$$

As the final step we construct the following functional equation which may be called a majorant (functional) equation to the equation (6.16):

$$\begin{aligned}
V(T,X) &= \frac{A}{(R-X)^K}T^K\\
&\quad +G_{K+1}\left(T,X,\frac{F}{R-X}V,\left\{\frac{F}{R-X}\frac{V}{T}\right\}_{j=1}^{d},\left\{\frac{10F}{(R-X)^2}V\right\}_{k=1}^{n}\right),
\end{aligned} \tag{6.23}$$

with $V(T,X) = O(T^K)$. The existence of unique formal solution $V(T,X)$ which is convergent follows from the classical implicit function theorem, and the above construction of the equation shows that $W(T,X) \ll V(T,X)$ which implies the convergence of $U(t,x)$. $\square$

7. Proof of Theorem 1.1, (ii).

We recall the equation we consider is given by

$$\left(\sum_{i,j=1}^{d} a_{ij}(x) t_i \partial_{t_j} + c(x)\right) v(t,x) = \sum_{|\alpha|=2} b_\alpha(x) t^\alpha + f_3(t,x,v,\partial_t v, \partial_x v), \tag{7.1}$$

where $c(x) = f_u(\mathbf{a}(x))$ with $c(0) \neq 0$ and $A(x) = (a_{ij}(x))_{ij}^d$ is a nilpotent matrix such that $A(x)^N \equiv \mathbf{O}$ but $A(x)^j \not\equiv \mathbf{O}$ for $0 \le j \le N-1$ $(1 \le N \le d)$.

We remark that by the assumption that $c(0) \neq 0$, we may assume $c(x) \equiv 1$ in the above equation by multiplying $c(x)^{-1}$ to the equation which does not change the assumption for $A(x)$.

Let assume the functions in the equation are holomorphic in x on $D_0(R)$ by an $R > 0$. Then we can easily examine the unique existence of the formal solution $v(t,x) = \sum_{|\alpha| \ge 2} v_\alpha(x) t^\alpha$ $(v_\alpha(x) \in \mathcal{O}(R))$. Indeed, under our assumptions the mapping

$$\sum_{i,j=1}^{d} a_{ij}(x) t_i \partial_{t_j} + 1 : \mathcal{O}(R)[t]_L \to \mathcal{O}(R)[t]_L,$$

is invertible by the fact that the matrix representation of the part of vector field which we set by $A(x)$ is nilpotent again. Therefore the formal solution is uniquely determined inductively on $L \ge 2$ for $v_L(t,x) = \sum_{|\alpha|=L} v_\alpha(x) t^\alpha \in \mathcal{O}(R)[t]_L$.

Our proof is thus reduced only to estimate the Gevrey order in t variables of the formal solution. Here we recall Lemma 4.1 which guarantees to make a change of variables t by $(\tau_1, \cdots, \tau_d) = (t_1, \cdots, t_d) Q(x)$ by $Q(x) \in GL(d, \mathcal{O}(R))$.

By the assumption of nilpotency for $A(x)$, there exists an invertible matrix $Q(x) = (Q_{ij}(x))$ over the field of meromorphic functions in a neighborhood of the origin such that

$$Q(x)^{-1}(a_{ij}(x))Q(x) = \mathrm{diag}(B_1, \cdots, B_I, O_J) \ : \ \text{Jordan canonical form}, \tag{7.2}$$

where diag$(\cdots)$ denotes the diagonal matrix with the diagonal blocks $(\cdots)$. Here, $B_i^{n_i} = \mathbf{O}$ $(n_i \ge 1)$ and O_J is the zero matrix block of size J with $n_1 + \cdots + n_I + J = d$, and by the assumption we have $\max\{n_1, \ldots, n_I\} = N$.

Now we make a "formal" change of variables by

$$(\tau_1, \ldots, \tau_d) = (t_1, \ldots, t_d) Q(x), \quad y_k = x_k \ (k = 1, \cdots, n).$$

Here the "formal" means that $Q(x)$ may admit meromorphic singular point at the origin, and it is an actual holomorphic change at the points if $Q(x)$ is holomorphically invertible at the origin.

Since $\partial_{t_i} = \sum_{j=1}^{d} Q_{ij}(x)\partial_{\tau_j}$ and $\partial_{x_k} = \sum_{j=1}^{d} t_i\{\partial_{x_k} Q_{ij}(x)\}\partial_{\tau_j} + \partial_{y_k}$, in the reduced equation by this change of variables the vector field is changed by the Jordan canonical form (7.2), and the nonlinear term f_3 is changed to g_3 which satisfies the same condition.

According to the form of (7.2), we make a further change of variables, $y \mapsto x \in \mathbf{C}^n$ (as before), and make a decomposition $\tau = (y, z) \in \mathbf{C}^d$ by

$$(y, z) = (\mathbf{y}^1, \dots, \mathbf{y}^I, z), \quad \mathbf{y}^i = (y_{i,1}, \dots, y_{i,n_i}) \in \mathbf{C}^{n_i}, \quad z = (z_1, \dots, z_J) \in \mathbf{C}^J$$

Now the equation (7.1) is reduced to the following equation:

(7.3)
$$\begin{cases} Pv(y, z, x) = \sum_{|\alpha|+|\beta|=2} \zeta_{\alpha\beta}(x) y^\alpha z^\beta + g_3(y, z, x, v, \partial_y v, \partial_z v, \partial_x v), \\ v(y, z, x) = O((|y| + |z|)^2), \end{cases}$$

where

$$P = \sum_{i=1}^{I} \sum_{j=1}^{n_i - 1} \delta \, y_{i,j+1} \partial_{y_{i,j}} + 1, \quad \delta \in \mathbf{C}, \tag{7.4}$$

$$\begin{aligned} & g_3(y, z, x, v, \zeta, \eta, \xi) \\ & \quad = \sum_{|\alpha|+|\beta|+2p+|q^1|+|q^2|+2|r| \geq 3} g_{\alpha\beta p q^1 q^2 r}(x) y^\alpha z^\beta v^p \zeta^{q^1} \eta^{q^2} \xi^r, \end{aligned} \tag{7.5}$$

where $q^1 \in \mathbf{N}^{n_1 + \cdots + n_I}$, $q^2 \in \mathbf{N}^J$.

We remark that the constant δ is assumed as small as we want by Remark 6.1.

Here we have to notice that in the reduced equation (7.3) the origin $x = 0$ may be a singular point. Therefore, the proof of the theorem is divided into two steps. In the first step, we prove the theorem under the assumption of holomorphy at $x = 0$. In the second step, we remove such restriction by using the maximum principle for the holomorphic functions from the fact that the equation has a unique formal solution $v(t, x) \in \mathcal{O}(R)[[t]]$ which was mentioned above.

7.1. **Holomorphic case.** The proof below follows the arguments in [S1] and [S3] by Shirai. Especially, in [S3], the case of absence of the variables x was studied, and some of proofs in the below will be omitted or shortened since they are essentially the same.

We assume the equation (7.3) is holomprhic in a neighborhood of the origin and we shall prove that the formal solution $v(y, z, x)$ of (7.3) belongs to $\mathcal{G}^{2N}$ in (y, z) variables with $N = \max\{n_i \; ; \; i = 1, 2, \cdots, I\}$. In order to do that it is sufficient to prove $v(y,z,x)$ belongs to some Gevrey space $\mathcal{G}^{\mathbf{s}}$ in (y,z) variables with $\mathbf{s}=(s_1,s_2,\cdots,s_d)$ such that $\|\mathbf{s}\|=\max\{s_j\}\le 2N$.

Let us prepare the following lemma:

Proposition 7.1. (i) *For all $L\ge 2$, there exists a radius $R>0$ independent of L such that the mapping $P:\mathcal{O}_0(R)[y,z]_L\longrightarrow \mathcal{O}_0(R)[y,z]_L$ is invertible.*

(ii) *Let $\widetilde{\mathbf{s}}=(\mathbf{s}_1,\cdots,\mathbf{s}_I,\mathbf{1}_J)\in\mathbf{N}^d$, where*

$$\mathbf{s}_i=(1,2,\cdots,n_i)\in\mathbf{N}^{n_i},\quad \mathbf{1}_J=(1,\cdots,1)\in\mathbf{N}^J,$$

as a manner corresponding to the decomposition $\tau=(y,z)$. For $\mathbf{k}_d=(k,\cdots,k)\in\mathbf{N}^d$ we define $\widetilde{\mathbf{s}}+\mathbf{k}_d$ (or $\widetilde{\mathbf{s}}+k$, for short) by the summation componentwisely.

For $f(y,z,x)\in\mathcal{O}_0(R)[y,z]_L$, if $\mathcal{B}^{\widetilde{\mathbf{s}}+k}(f)(y,z,x)\ll W_L(X)T^L$ ($T=|y|+|z|,X=|x|$), then there exists a positive constant $C>0$ independent of L such that

$$\mathcal{B}^{\widetilde{\mathbf{s}}+k}(P^{-1}f)(y,z,x)\ll C\,W_L(X)T^L. \tag{7.6}$$

Proof. (i) is obvious since the vector field is nilpotent as we mentioned before.

(ii) For an operator Q on $\mathbf{C}[t]_L$, $\|Q\|_{\widetilde{\mathbf{s}}+k}$ denotes the operator norm equipped with the norm $\|\cdot\|_{\widetilde{\mathbf{s}}+k}$ on $\mathbf{C}[t]_L$. Then it is easily proved that

$$\|y_{i,j+1}\partial_{y_{i,j}}\|_{\widetilde{\mathbf{s}}+k}\le 1$$

for all i and j. Therefore by taking $|\delta|$ so small that $(d-J-I)|\delta|<1$ we get the majorant estimate (7.6) by $C=1/(1-(d-J-I)|\delta|)$. $\square$

Remark 7.1. This lemma shows the bijectivity of the mapping $P:\mathcal{G}^{\widetilde{\mathbf{s}}+k}\to\mathcal{G}^{\widetilde{\mathbf{s}}+k}$ for all $k\ge 0$. Indeed, let $f(y,z,x)=\sum_{L\ge1}f_L(y,z,x)\in\mathcal{G}^{\widetilde{\mathbf{s}}+k}$ with $f_L(y,z,x)\in\mathcal{O}_0(R)[y,z]_L$. Since $\mathcal{B}^{\widetilde{\mathbf{s}}+k}f(y,z,x)=\sum_{L\ge1}\mathcal{B}^{\widetilde{\mathbf{s}}+k}f_L(y,z,x)\in\mathcal{O}_{y,z,x}$, there exist positive constants M and R' such that

$$\mathcal{B}^{\widetilde{\mathbf{s}}+k}f(y,z,x)\ll\frac{M}{(1-X/R')(1-T/R')}=\frac{M}{1-X/R'}\sum_{L\ge1}\frac{T^L}{R'^L},$$

where T and X are given as above. This means that

$$\mathcal{B}^{\widetilde{\mathbf{s}}+k}f_L(y,z,x)\ll\frac{MT^L}{R'^L(1-X/R')},$$

and for the formal inverse $P^{-1}f$ we have

$$\mathcal{B}^{\widetilde{\mathbf{s}}+k}(P^{-1}f)(y,z,x)\ll\frac{CM}{(1-X/R')(1-T/R')}\in\mathcal{O}_{y,z,x}.$$

We put $U(y,z,x) = Pv(y,z,x)$ as a new unknown function. Then, $U(y,z,x)$ satisfies the following equation:

(7.7)

$$\begin{cases} U(y,z,x) = \sum_{|\alpha|+|\beta|=2} \zeta_{\alpha\beta}(x) y^\alpha z^\beta \\ \qquad\qquad\qquad + g_3(y,z,x,P^{-1}U, \partial_y P^{-1}U, \partial_z P^{-1}U, \partial_x P^{-1}U), \\ U(y,z,x) = O((|y|+|z|)^2). \end{cases}$$

Now we apply the $\widetilde{\mathbf{s}}$-Borel transform to the equation (7.7), we obtain

$$\begin{aligned} \mathcal{B}^{\widetilde{\mathbf{s}}}(U)(y,z,x) \quad = \quad & \sum_{|\alpha|+|\beta|=2} \zeta_{\alpha\beta}(x) \frac{(|\alpha|+|\beta|)!}{(\widetilde{\mathbf{s}}\cdot(\alpha,\beta))!} y^\alpha z^\beta \\ & + \mathcal{B}^{\widetilde{\mathbf{s}}}\{g_3(y,z,x,P^{-1}U, \partial_y P^{-1}U, \partial_z P^{-1}U, \partial_x P^{-1}U)\}. \end{aligned} \tag{7.8}$$

In order to construct a majorant equation for (7.8), we prepare the following lemma:

Lemma 7.1. (i) *The Borel transform of a product $(uv)(y,z,x)$ is majorized by*

$$\mathcal{B}^{\widetilde{\mathbf{s}}}(uv)(y,z,x) \ll N\mathcal{B}^{\widetilde{\mathbf{s}}}(|u|)(y,z,x) \times \mathcal{B}^{\widetilde{\mathbf{s}}}(|v|)(y,z,x), \tag{7.9}$$

where $N = \max\{n_1, \dots, n_I\}$.

(ii) *If $\mathcal{B}^{\widetilde{\mathbf{s}}}(u)(y,z,x) \ll W(T,X)$ $(T = |y|+|z|, X = |x|)$, then there exists a positive constant $C_1 > 0$ independent of y, z and x such that the Borel transforms of $\partial_{y_{i,j}} u$, $\partial_{z_k} u$ and $\partial_{x_k} u$ are majorized by*

$$\mathcal{B}^{\widetilde{\mathbf{s}}}(\partial_{y_{i,j}} u)(y,z,x) \ll C_1 \partial_T (T\partial_T)^{j-1} W(T,X), \tag{7.10}$$

$$\mathcal{B}^{\widetilde{\mathbf{s}}}(\partial_{z_k} u)(y,z,x) \ll C_1 \partial_T W(T,X), \tag{7.11}$$

$$\mathcal{B}^{\widetilde{\mathbf{s}}}(\partial_{x_k} u)(y,z,x) \ll C_1 \partial_X W(T,X), \tag{7.12}$$

Proof. The proofs are the same with those of Lemma 2 in [S3], so we omit it. □

Now we consider the following equation which is a majorant equation of (7.8):

$$\begin{aligned} W(T,X) = & \left(\sum_{|\alpha|+|\beta|=2} |\zeta_{\alpha\beta}|(\mathbf{X}) \frac{(|\alpha|+|\beta|)!}{(\widetilde{\mathbf{s}}\cdot(\alpha,\beta))!} \right) T^2 \\ & + |g_3| \left(\mathbf{T}, \mathbf{X}, C'W, \left\{ \left\{ C'\partial_T (T\partial_T)^{j-1} W \right\}_{j=1}^{n_i} \right\}_{i=1}^{I}, \right. \\ & \qquad \left. \{C'\partial_T W\}_{k=1}^{J}, \{C'\partial_X W\}_{k=1}^{n} \right), \end{aligned} \tag{7.13}$$

with $W(T,X) = O(T^2)$ where $\mathbf{T} = (T,\dots,T) \in \mathbf{C}^d$, $\mathbf{X} = (X,\dots,X) \in \mathbf{C}^n$ and $C' = C_1 C N$.

Now by the construction of the equation (7.13), we easily see that the formal solution $W(T,X) \in \mathcal{O}_X[[T]]$ is a majorant function of $\mathcal{B}^{\widetilde{\mathbf{s}}}(U)(y,z,x)$ of (7.8) by replacing $T = y_{1,1} + \cdots + y_{I,n_I} + z_1 + \cdots + z_J$ and $X = x_1 + \cdots + x_n$.

Here we recall the result in [S1] by Shirai in a special form attached to our case. Let us consider the following equation.

$$V(T,X) = g(X)T^K + h_{K+1}(T,X,V,\{D_T^j V\}_{j=1}^p, D_X V)$$

with $V = O(T^K)$, where $g(X)$ and $h_{K+1}(T,X,V,\tau,\xi)$ $(\tau \in \mathbf{C}^p, \xi \in \mathbf{C})$ are holomorphic in a neighborhood of the origin and

$$h_{K+1}(T,X,V,\tau,\xi) = {\sum}' h_{ab\{c(j)\}d}(X) T^a V^b \prod_{j=1}^{p} \tau_j^{c(j)}\, \xi^d,$$

and the summation $\sum'$ is taken over

$$V(a,b,\{c(j)\},d) := a + Kb + \sum_j (K-j)c(j) + Kd \geq K+1,$$

the left hand side means the order of zeros in T of each monomial by substituting $V(t,x) = O(T^K)$.

Then the formal solution $V(T,X) \in \mathcal{O}_X[[t]]$ which exists uniquely belongs to $\mathcal{G}^{\sigma+1}$ in T variable with

$$\sigma = \max\left\{ \frac{A(a,b,\{c(j)\},d)}{V(a,b,\{c(j)\},d) - K} \ ;\ h_{ab\{c(j)\}d}(x) \not\equiv 0 \right\},$$

by $A(a,b,\{c(j)\},d)$ $(\in \{0,1,2,\cdots,p\})$ which denotes the maximal order of differentiations which appears in the monomial. (This is a special case of Theorem 1 in [S1].)

We return to the equation (7.13). In this case, $K = 2$, $V(a,b,\{c(j)\},d) - K \geq 1$ and $A(a,b,\{c(j)\},d) \leq \max\{n_i \ ;\ i = 1,2,\cdots,I\} = N$ which shows that $W(T,X) \in \mathcal{G}^{N+1}$ in T variable. Therefore $\mathcal{B}^{\widetilde{\mathbf{s}}}(U)$ $(U = Pv)$ belongs to the Gevrey space $\mathcal{G}^{N+1}$ in τ variables $\tau(= (y,z))$ variables, which implies $U = Pv \in \mathcal{G}^{\widetilde{\mathbf{s}}+N}$ in τ variables, and hence $v(\tau,x) = P^{-1}U \in \mathcal{G}^{\widetilde{\mathbf{s}}+N}$ in τ variables by Proposition 7.1 and Remark 7.1. Then by Lemma 4.1, we have $v(t,x) \in \mathcal{G}^{2N}$ in t variables, since each component of $\widetilde{\mathbf{s}}$ is estimated by $N = \max\{n_i\,;\, i = 1,2,\cdots,I\}$. $\square$

7.2. **Meromorphic case.** In this subsection, we shall prove the theorem in the case where $Q(x)$ or $Q(x)^{-1}$ is singular at the origin by the idea used in [M] by Miyake where the inverse theorem of Cauchy-Kowalevski's theorem for general systems was studied. The theorem is an immediate result from the following lemma:

Lemma 7.2. *Assume that $Q(x)$ or $Q(x)^{-1}$ is singular at the origin. We may assume that $Q(x)$ and $Q(x)^{-1}$ are holomorphic on $\prod_{j=1}^{n}\{R_j-\varepsilon \le |x_j| \le R_j+\varepsilon\} \subset D_0(R)$ by suitable taking positive constants $R_j > 0$ and $\varepsilon > 0$ ($j = 1, 2, \cdots, n$) such that $0 < R_j - \varepsilon < R_j + \varepsilon < R$. Then the formal solution $v(\tau, x)$ ($\tau = (y, z)$) of (7.3) belongs to $\mathcal{G}^{2N}$ in τ variables on $\prod_{j=1}^{n}\{|x_j| \le R_j\}$.*

Proof. We, first, notice that we already know there exists a unique formal solution $v(\tau, x) = \sum_{|\alpha|\ge 2} v_\alpha(x)t^\alpha \in \mathcal{O}_x[[t]]$, where we may assume that $v_\alpha(x) \in \mathcal{O}_0(R)$ by a small $R > 0$ for all α. We may consider that this R is the one in the statement of the lemma.

Let $\widehat{x} = (\widehat{x}_1, \cdots, \widehat{x}_n) \in \prod_{j=1}^{n}\{|x_j| = R_j\}$ be arbitrary fixed. Then by the assumption, $Q(x)$ is holomorphically invertible on ε neighborhood of $\widehat{x}$. By the result in the previous subsection, we know that the formal solution $v(\tau, x)$ belongs to $\mathcal{G}^{2N}$ in τ variables in a neighborhood of $\widehat{x}$. Therefore there exists a positive constant $r(\widehat{x})$ (which may depend on $\widehat{x}$) such that the following Gevrey estimates hold by positive constants $A_{\widehat{x}}$ and $B_{\widehat{x}}$ which may depend on $\widehat{x}$.

$$\max_{|x_j-\widehat{x}_j|\le r(\widehat{x})} |v_\alpha(x)| \le A_{\widehat{x}}\, B_{\widehat{x}}^{|\alpha|}\, \{(2N-1)|\alpha|\}!, \tag{7.14}$$

for all $\alpha \in \mathbf{N}^d$ with $|\alpha| \ge 2$.

Since the polycircle $C(R) = \prod\{|x_j| = R_j\}$ ($R = (R_1, \cdots, R_d)$) is compact, we can take finite number of $\{\widehat{x}^{(k)}\}_k$ on the polycircle so that the union of $r(\widehat{x}^{(k)})$ neighborhood of $\widehat{x}^{(k)}$'s covers the polycircle $C(R)$. Now by taking A the maximum of $A_{\widehat{x}^{(k)}}$'s and B the maximum of $B_{\widehat{x}^{(k)}}$'s, we get the following Gevrey estimates on the polycircle $C(R)$,

$$\max_{x\in C(R)} |v_\alpha(x)| \le A\, B^{|\alpha|}\, \{(2N-1)|\alpha|\}!, \tag{7.15}$$

for all $\alpha \in \mathbf{N}^d$ with $|\alpha| \ge 2$. Since $v_\alpha(x)$ are all holomorphic on $D_0(R)$, by the maximum principle we get the same Gevrey estimation on the polydisc $\prod_j\{|x_j| \le R_j\}$, which proves the lemma. □

8. Appendix: Alternative Proof of Theorem 5.1.

In this section, we give an alternative proof of Theorem 5.1. The proof given in Section 6 is based on the view point as evolution equations in t variables, and that given in this Appendix is based on the view point that the role of variables t and x are equivalent in a sense.

By a linear change of t variables which brings $(a_{ij}(0))$ to its Jordan canonical form, the equation (5.4) is reduced to the following equation:

$$
\begin{aligned}
(8.1) \qquad (\Lambda+\Delta)w(t,x) &= \sum_{i,j=1}^{d} \alpha_{ij}(x)t_i\partial_{t_j}w + \eta(x)w + \sum_{|\alpha|=K}\zeta_\alpha(x)t^\alpha \\
&\quad + g_{K+1}(t,x,w,\partial_t w,\partial_x w),
\end{aligned}
$$

where

$$
(8.2) \qquad \Lambda = \sum_{j=1}^{d}\lambda_j t_j\partial_{t_j} + c(0), \qquad \Delta = \sum_{j=1}^{d-1}\delta_j t_{j+1}\partial_{t_j},
$$

and $\alpha_{ij}(x)\,(=O(|x|))$, $\eta(x)\,(=O(|x|))$ are holomorphic in a neighborhood of the origin, and g_{K+1} is holomorphic in a neighborhood of the origin with a similar Taylor expansion with (5.5).

Let $\mathbf{C}[t]_L[x]_M$ be a set of homogeneous polynomials of degree L in t and of degree M in x, that is,

$$
\mathbf{C}[t]_L[x]_M = \left\{ u_{LM}(t,x) = \sum\nolimits_{|\alpha|=L,|\beta|=M} u_{\alpha\beta}t^\alpha x^\beta \ ;\ u_{\alpha\beta}\in\mathbf{C} \right\}.
$$

We define a set of homogeneous polynomials of degree L in t by

$$
\mathbf{C}[t]_L[[x]] = \left\{ u_L(t,x) = \sum\nolimits_{M\ge 0} u_{LM}(t,x) \ ;\ u_{LM}(t,x)\in\mathbf{C}[t]_L[x]_M \right\}.
$$

By substituting $w(t,x)=\sum_{L\ge K} w_L(t,x)$ ($w_L(t,x)\in\mathbf{C}[t]_L[[x]]$) into (8.1), we get the following recursion formula:

$$
(\Lambda+\Delta)w_K(t,x) = \sum_{i,j=1}^{d}\alpha_{ij}(x)t_i\partial_{t_j}w_K(t,x) + \eta(x)w_K(t,x) + \sum_{|\alpha|=K}\zeta_\alpha(x)t^\alpha,
$$

and for $L\ge K+1$,

$$
\begin{aligned}
(\Lambda+\Delta)w_L(t,x) &= \sum_{i,j=1}^{d}\alpha_{ij}(x)t_i\partial_{t_j}w_L(t,x) + \eta(x)w_L(t,x) \\
&\quad + H_L(t,x,\{w_{L'}\}_{L'<L},\{\partial_t w_{L'}\}_{L'<L},\{\partial_x w_{L'}\}_{L'<L}),
\end{aligned}
$$

where H_L denotes a homogeneous polynomial of degree L in t.

Next we substitute $w_L(t,x)=\sum_{M\ge 0} w_{LM}(t,x)$ ($w_{LM}(t,x)\in\mathbf{C}[t]_L[x]_M$) into the above recursion formulas, we have

$$
(8.3) \qquad (\Lambda+\Delta)w_{LM}(t,x) = H_{LM}(t,x,\{w_{L'M'}(t,x)\}_{(L',M')<(L,M)}),
$$

where H_{LM} denotes a homogeneous polynomial of degree L in t and of degree M in x, and $(L',M')<(L,M)$ denotes the lexicographic order.

This recursion formula (8.3) has a unique solution $w_{LM}(t,x)$ by the following Lemma:

Lemma 8.1. *Let $P = \Lambda + \Delta$. Then we have:*

(i) *The mapping $P : \mathbf{C}[t]_L[x]_M \longrightarrow \mathbf{C}[t]_L[x]_M$ is invertible for all $L \geq K$ and $M \geq 0$.*

(ii) *If a majorant relation $u_{LM}(t,x) \ll W_{LM} \times T^L X^M$ ($T = t_1 + \cdots + t_d, X = x_1 + \cdots + x_n$) holds, then there exists a positive constant C_1 independent of L and M such that*

(8.4)
$$P^{-1}u_{LM}(t,x) \ll \frac{C_1}{L} W_{LM} \times T^L X^M = C_1\, W_{LM} \times (T\partial_T)^{-1} T^L X^M.$$

Proof. (i) The invertibility of Λ follows by the modified Poincaré condition (5.3). Since the matrix representation of Λ^{-1} is diagonal, $\Lambda^{-1}\Delta$ is nilpotent again. These observations imply the invertibility of $P = \Lambda + \Delta$.

(ii) We introduce a norm of $u_{LM}(t,x) \in \mathbf{C}[t]_L[x]_M$ by

$$||u_{LM}|| := \inf\{C > 0\ ;\ u_{LM}(t,x) \ll CT^L X^M\}. \tag{8.5}$$

Then by the condition (5.3), the operator norm of the inverse Λ^{-1} is estimated by $||\Lambda^{-1}|| \leq 1/(C_0 L)$ where C_0 is the constant in (5.3). Furthermore, we can estimate the operator norm of Δ by $||\Delta|| \leq \max\{|\delta_1|, \cdots, |\delta_{d-1}|\}L$. Here we recall that we may assume that $|\delta_j|$ are as small as we want. Therefore we may assume that $||\Lambda^{-1}\Delta|| \leq (C_0)^{-1} \max\{|\delta_1|, \cdots, |\delta_{d-1}|\} \leq 1/2$. By this choice of $\{\delta_j\}$, the operator norm of $P^{-1} = (I - \Lambda^{-1}\Delta)^{-1}\Lambda^{-1}$ is estimated by

$$||P^{-1}|| \leq \frac{||\Lambda^{-1}||}{1 - ||\Lambda^{-1}\Delta||} \leq \frac{2}{C_0 L}.$$

Therefore, it is enough to take $C_1 = 2/C_0$. □

Next, we shall prove the convergence of the formal solution $w(t,x)$. We put $U(t,x) = Pw(t,x)$ as a new unknown function. Then $U(t,x)$ satisfies the following equation:

$$\begin{aligned} &\left\{I - \textstyle\sum_{i,j=1}^d \alpha_{ij}(x) t_i \partial_{t_j} P^{-1} - \eta(x) P^{-1}\right\} U \\ &\qquad = \sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha + g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U). \end{aligned} \tag{8.6}$$

By Lemma 8.1, (ii), if a majorant relation $U(t,x) \ll W(T,X)$ holds, then we have the following majorant relations:

- $\sum_{i,j=1}^d \alpha_{ij}(x) t_i \partial_{t_j} P^{-1} U \ll C_1 \left(\sum_{i,j=1}^d |\alpha_{ij}|(\mathbf{X})\right) W$,
- $\eta(x) P^{-1} U \ll C_1 |\eta|(\mathbf{X}) W$,
- $\sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha \ll \left(\sum_{|\alpha|=K} |\zeta_\alpha|(\mathbf{X})\right) T^K$,
- $g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U)$

$$\ll |g_{K+1}|(\mathbf{T}, \mathbf{X}, C_1 W, C_1 W/T, C_1 \partial_X (T\partial_T)^{-1} W),$$

where $\mathbf{T} = (T, T, \dots, T) \in \mathbf{C}^d$ and $\mathbf{X} = (X, X, \dots, X) \in \mathbf{C}^n$. Here $|f|(x)$ and $|g_{K+1}|(t, x, u, \tau, \xi, \eta)$ are defined as follows:

For a formal power series $f(x) = \sum f_\alpha x^\alpha$, we define $|f|(x) = \sum |f_\alpha| x^\alpha$, and for $g_{K+1}(t, x, u, \tau, \xi) = \sum g_{\alpha pqr}(x) t^\alpha u^p \tau^q \xi^r$, we define

$$|g_{K+1}|(t, x, u, \tau, \xi) = \sum |g_{\alpha pqr}|(x) t^\alpha u^p \tau^q \xi^r.$$

Let us consider the following equation which is a majorant equation for the equation (8.6):

(8.7)

$$\begin{cases} W(T,X) = Z(X)T^K \\ \qquad\qquad +G_{K+1}\left(T, X, C_1 W, \left\{\dfrac{C_1 W}{T}\right\}, \left\{C_1 \partial_X (T\partial_T)^{-1} W\right\}\right), \\ W(T,X) = O(T^K), \end{cases}$$

where $Z(X) = P(X) \sum_{|\alpha|=K} |\zeta_\alpha|(\mathbf{X})$ ($P(X)$ is defined below), and

$$G_{K+1}(T, X, u, \tau, \xi) = P(X)|g_{K+1}|(\mathbf{T}, \mathbf{X}, u, \tau, \xi)$$

with

$$P(X) = \left(1 - C_1 \textstyle\sum_{i,j=1}^d |\alpha_{ij}|(\mathbf{X}) - C_1 |\eta|(\mathbf{X})\right)^{-1} \in \mathbf{C}\{X\}.$$

By this construction of the equation, it is easily examined that $U(t,x) \ll W(T,X)$.

We take majorant functions of $Z(X)$ and G_{K+1} by

$$Z(X) \ll \frac{A}{(R-X)^K} =: Q(X), \tag{8.8}$$

$$\begin{aligned} &G_{K+1}(T, X, u, \tau, \xi) \\ &\ll \sum_{|\alpha|+Kp+(K-1)|q|+K|r| \geq K+1} \frac{G_{\alpha pqr}}{(R-X)^{|\alpha|+p+|q|+|r|}} T^{|\alpha|} u^p \tau^q \xi^r \\ &=: R_{K+1}(T, X, u, \tau, \xi). \end{aligned} \tag{8.9}$$

Then we consider the following equation.

(8.10)

$$\begin{cases} V(T,X) = Q(X)T^K \\ \qquad\qquad +R_{K+1}\left(T, X, C_1 V, \left\{\dfrac{C_1 V}{T}\right\}, \left\{C_1 \partial_X (T\partial_T)^{-1} V\right\}\right), \\ V(T,X) = O(T^K). \end{cases}$$

By this construction of the majorant equation (8.10), we have

$$U(t,x) \ll W(T,X) \ll V(T,X).$$

We substitute $V(T,X) = \sum_{L\geq K} V_L(X)T^L$ into (8.10). Then we obtain the following recurrent formulas:

$$V_K(X) = \frac{A}{(R-X)^K} = Q(X), \tag{8.11}$$

and for $L \geq K+1$

$$\begin{aligned} V_L(X) \; &= \sum_{|\alpha|+Kp+(K-1)|q|+K|r|\geq K+1} \frac{G_{\alpha pqr} C_1^{p+|q|+|r|}}{(R-X)^{|\alpha|+p+|q|+|r|}} \\ &\quad \times {\sum}' \prod_{l=1}^{p} V_{L_l}(X) \prod_{l=1}^{|q|} V_{M_l}(X) \prod_{l=1}^{|r|} N_l^{-1} \partial_X V_{N_l}(X), \end{aligned} \tag{8.12}$$

where the summation $\sum'$ is taken over

$$|\alpha| + \sum_{l=1}^{p} L_l + \sum_{l=1}^{|q|} (M_l - 1) + \sum_{l=1}^{|r|} N_l = L. \tag{8.13}$$

By using these formulas we can prove the following lemma.

Lemma 8.2. *The coefficients* $\{V_L(X)\}_{L\geq K}$ *are given by*

$$V_L(X) = \sum_{j=K}^{7L-6K} \frac{V_{L,j}}{(R-X)^j}, \quad L \geq K. \tag{8.14}$$

The proof is done by the same way with Lemma 6.1, so we omit it.

Thus we see that the formal solution $V(T,X)$ of (8.10) is written by

$$V(T,X) = \sum_{L\geq K} \sum_{j=K}^{7L-6K} \frac{V_{L,j}}{(R-X)^j} T^L,$$

and further we see that $\partial_X (T\partial_T)^{-1} V(T,X)$ is majorized by

$$\partial_X (T\partial_T)^{-1} V(T,X) = \sum_{L\geq K} \sum_{j=K}^{7L-6K} \frac{j}{L} \frac{V_{L,j}}{(R-X)^{j+1}} T^L \ll \frac{7}{R-X} V(T,X).$$

We finally obtain the following majorant functional equation:

$$Y = \frac{A}{(R-X)^K} T^K + R_{K+1}\left(T, X, C_1 Y, \left\{\frac{C_1 Y}{T}\right\}, \left\{\frac{7C_1}{R-X} Y\right\}\right), \tag{8.15}$$

with $Y = Y(T,X) = O(T^K)$. By these procedures of constructing the equation (8.15), we get the following majorant relations between the uniquely determined formal solutions.

$$Y(T,X) \gg V(T,X) \gg W(T,X) \gg U(t,x).$$

At the end, we notice that the convergent of $Y(T, X)$ follows from the classical implicit function theorem which assures the unique existence of convergent solution of (8.15).

This completes the second proof. □

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Masatake Miyake
Graduate School of Mathematics,
Nagoya University
mmiyake@math.nagoya-u.ac.jp

Akira Shirai
Graduate School of Mathematics,
Nagoya University
m96034q@math.nagoya-u.ac.jp