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Staggered Runge-Kutta Schemes for Semilinear Wave Equations

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Abstract

A staggered Runge-Kutta (staggered RK) scheme is the time integration Runge-Kutta type scheme based on staggered grid, which was proposed by Ghrist and Fornberg and Driscoll in 2000. Afterwords, Vewer presented efficiency of the scheme for linear and semilinear wave equations through numerical experiments. We study stability and convergence properties of this scheme for semilinear wave equations. In particular, we prove convergence of a fully discrete scheme obtained by applying the staggered RK scheme to the MOL approximation of the equation.

Key words: Wave equations, Explicit time integration, Staggered Runge-Kutta schemes, Convergence, Stability Analysis

1. Introduction

We consider initial-boundary value problems of the form

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = D\Delta u + g(t, x, u), \quad 0 \le t \le T, \quad x \in \Omega, \\
&\Phi_b u = \varphi(t, x), \quad 0 \le t \le T, \quad x \in \partial\Omega, \\
&u(0, x) = u_0(x), \quad \frac{\partial u}{\partial t}(0, x) = v_0(x), \quad x \in \Omega.
\end{aligned}$$

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Here $u(t,x)$ is an $\mathbb{R}$-valued unknown function, Ω is a bounded domain in $\mathbb{R}^i, i = 1,2,3$ with the boundary $\partial\Omega$, Δ is the Laplace operator, D is a positive constant, and $g(x,t,u)$ is an $\mathbb{R}$-valued given function. Also, Φ_b is a boundary operator and $u_0(x),\ v_0(x),\ \varphi(t,x)$ are given functions.
Many important wave equations, such as the Klein-Gordon equation (see, e.g., [10], [19]) and the nonlinear Klein-Gordon equation (see [17]), are represented in this form. To apply numerical schemes, we may use the form

$$\begin{aligned}&\frac{\partial u}{\partial t} = v, \quad \frac{\partial v}{\partial t} = D\Delta u + g(t,x,u), \quad 0 \le t \le T, \quad x \in \Omega,\\ &\Phi_b u = \varphi(t,x), \quad 0 \le t \le T, \quad x \in \partial\Omega,\\ &u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \Omega.\end{aligned} \tag{1}$$

A well-known approach in the numerical solution of wave problems in partial differential equations (PDEs) is the method of lines (MOL) (see [12]). In this approach, PDEs are first discretized in space by finite difference or finite element techniques to be converted into a system of ordinary differential equations (ODEs). Let $\Omega_h \subset \bar{\Omega}$ be a grid with mesh width $h > 0$, and $\boldsymbol{V}_h$ be the vector space of all functions from Ω_h to $\mathbb{R}$. An MOL approximation of (1) is written in the form

$$\frac{du_h(t)}{dt} = v_h(t), \quad \frac{dv_h(t)}{dt} = DL_h u_h(t) + \varphi_h(t) + g_h(t, u_h(t)). \tag{2}$$

Here $u_h,\ v_h$ are approximation functions of u and v such that $u_h(t),\ v_h(t) \in \boldsymbol{V}_h$ for $t \in [0,T]$, L_h is a difference approximation of Δ, g_h is a function from $[0,T]\times\boldsymbol{V}_h$ to $\boldsymbol{V}_h$ defined by $g_h(t,u_h)(x) = g(t,x,u_h(t)),\ x \in \Omega_h$, for $t \in [0,T]$, $u_h \in \boldsymbol{V}_h$, and $\varphi_h(t)$ is a function determined from the boundary condition.
In order to get the stable numerical solution of (2), Ghrist et al. introduced time-staggered schemes which is based on the idea of the staggered grid. The staggered grid is used to get explicit stable schemes in many fields. For example, the FDTD scheme (see [18]) in the electromagnetic field analysis and the SMAC scheme (see, e.g., [3], [9]) in the fluid calculation use staggered grid in space discretization. To the contrary, Ghrist et al. [5] consider staggered grid in time discretization and introduced the staggered Runge-Kutta (staggered RK) schemes. In particular, they proposed a forth-order, explicit, staggered RK scheme (RKS4) and studied stability and convergence of staggered RK schemes applied to ODEs. Vewer. (see, [15], [16]) presented efficiency of RKS4 for linear and semilinear wave equations through numerical experiments. As is well known, RK approximations for PDEs suffer from

order reduction phenomena. That is, the order of time-stepping in the fully discrete scheme is, in general, less than that of the underlying RK scheme (see, e.g., [8], [11], [14] on order reduction phenomena of RK schemes in the PDE context). Vewer observes the order of RKS4 is three, while that of the classical RK scheme is two. He also gives an analysis of this phenomenon.

In this paper, we study stability and convergence of staggered RK schemes for (2). Specifically, we introduce a new stability condition which guarantees the boundedness of numerical solutions and prove convergence of the schemes.

The paper is organized as follows. In the next section (Section 2), we introduce some notation, including the form of the staggered RK schemes. In Section 3, we prove a theorem which describes the boundedness of the numerical solution. In Section 4, we prove a theorem which describes convergence of the scheme applied to (2). In Section 5, we estimate the order of convergence by using a numerical experiment.

2. Preliminaries

Let $\tau > 0$ be a step size. We define the step points $t_n = n\tau$, $t_{n+1/2} = (n+1/2)\tau$ for integer $n \geq 0$.

As [5], for positive integer s, a staggered RK scheme for ODEs of the form

$$\begin{cases} u' &= f(t,v) \\ v' &= g(t,u) \end{cases}, \quad 0 \leq t \leq T, \ u, v \in \mathbb{R} \tag{3}$$

is given as

$$
\begin{aligned}
v_{n+1/2,1} &= v_{n+1/2}, \\
u_{n,i} &= u_n + \tau \sum_{j=1}^{i} b_{i,j} f(t_{n+1/2} + e_j \tau, v_{n+1/2,j}),\ i = 1, \cdots, s-1, \\
v_{n+1/2,i} &= v_{n+1/2} + \tau \sum_{j=1}^{i-1} a_{i,j} g(t_n + c_j \tau, u_{n,j}),\ i = 2, \cdots, s, \\
u_{n+1} &= u_n + \tau \sum_{i=1}^{s} d_i f(t_{n+1/2} + e_i \tau, v_{n+1/2,i}),
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
u'_{n+1,1} &= u_{n+1}, \\
v'_{n+1/2,i} &= v_{n+1/2} + \tau \sum_{j=1}^{i} b'_{i,j} g(t_{n+1} + e'_j \tau, u'_{n+1,j}),\ i = 1, \cdots, s-1, \\
u'_{n+1,i} &= u_{n+1} + \tau \sum_{j=1}^{i-1} a'_{i,j} f(t_{n+1/2} + c'_j \tau, v'_{n+1/2,j}),\ i = 2, \cdots, s, \\
v_{n+3/2} &= v_{n+1/2} + \tau \sum_{i=1}^{s} d'_i g(t_{n+1} + e'_i \tau, u'_{n+1,i})
\end{aligned}
\tag{5}
$$

and the abscissae

$$
\begin{aligned}
c_i &= \sum_{j=1}^{i} b_{i,j},\ c'_i = \sum_{j=1}^{i} b'_{i,j},\ i = 1, \ldots, s-1, \\
e_i &= \sum_{j=1}^{i-1} a_{i,j},\ e'_i = \sum_{j=1}^{i-1} a'_{i,j},\ i = 2, \ldots, s.
\end{aligned}
\tag{6}
$$

Here $a_{i,j},\ b_{i,j},\ a'_{i,j},\ b'_{i,j},\ c_i,\ c'_i,\ d_i,\ d'_i,\ e_i,\ e'_i$ are coefficients, $e_1 = e'_1 = 0$, $u_{n,i},\ v_{n+1/2,i},\ u'_{n+1,i},\ v'_{n+1/2,i}$ are intermediate variables, u_n and $v_{n+1/2}$ are approximate values of $u(t_n)$ and $v(t_{n+1/2})$, respectively.
We describe the algorithm of the staggered RK scheme. In the first step, we calculate u_1 from u_0 and $v_{1/2}$ by (4), where $v_{1/2}$ is produced by given initial values $u_0(x) = u_0,\ v_0(x) = v_0,\ x \in \Omega_h$ and using the Runge-Kutta scheme. The next step, we calculate $v_{3/2}$ from $v_{1/2}$ and u_1 by (5). By this way, we calculate u_{n+1} from u_n and $v_{n+1/2}$ by (4), and $v_{n+3/2}$ from $v_{n+1/2}$ and u_{n+1}

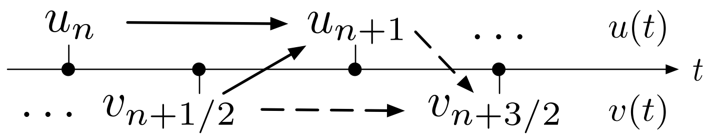

Figure 1: The approximation values of staggered RK schemes

by (5).
Fig.1 shows this process. The solid arrow describes the process of calculating u_{n+1} and the dashed arrow describes the process of calculating $v_{n+3/2}$. All the approximate values are calculated explicitly.
We introduce some notation. The $m \times m$ identity matrix will be denoted by I_m. We use the standard symbol $\mathbf{1} = (1, \cdots, 1)^T \in \mathbb{R}^s$.
To estimate stability of the scheme, we use the following linear test equation:

$$\begin{cases} u'(t) = v(t) \\ v'(t) = -\omega^2 u(t) \end{cases}, \quad \omega \in \mathbb{R}_{\geq 0} \tag{7}$$

with $\mathbb{R}_{\geq 0} = \{x; x \geq 0, x \in \mathbb{R}\}$.
Applying (4)-(5) to (7), we get

$$\begin{aligned} V_{n+1/2} &= \mathbf{1} v_{n+1/2} - \tau\omega^2 A U_n, \\ U_n &= \mathbf{1} u_n + \tau B V_{n+1/2}, \\ u_{n+1} &= u_n + \tau d V_{n+1/2}, \\ U'_{n+1} &= \mathbf{1} u_{n+1} + \tau A' V'_{n+1/2}, \\ V'_{n+1/2} &= \mathbf{1} v_{n+1/2} - \tau\omega^2 B' U'_{n+1}, \\ v_{n+3/2} &= v_{n+1/2} - \tau\omega^2 d' U'_{n+1}, \end{aligned} \tag{8}$$

where

$$A = \begin{pmatrix} 0 & & & \\ a_{2,1} & 0 & O & \\ \vdots & \ddots & \ddots & \\ a_{s,1} & \cdots & a_{s,s-1} & 0 \end{pmatrix}, \ B = \begin{pmatrix} b_{1,1} & & & \\ b_{2,1} & b_{2,2} & O & \\ \vdots & \vdots & \ddots & \\ b_{s,1} & b_{s,2} & \cdots & b_{s,s} \end{pmatrix}, \ d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_s \end{pmatrix}^T,$$

$$A' = \begin{pmatrix} 0 & & & \\ a'_{2,1} & 0 & O & \\ \vdots & \ddots & \ddots & \\ a'_{s,1} & \cdots & a'_{s,s-1} & 0 \end{pmatrix}, \ B' = \begin{pmatrix} b'_{1,1} & & & \\ b'_{2,1} & b'_{2,2} & O & \\ \vdots & \vdots & \ddots & \\ b'_{s,1} & b'_{s,2} & \cdots & b'_{s,s} \end{pmatrix}, \ d = \begin{pmatrix} d'_1 \\ d'_2 \\ \vdots \\ d'_s \end{pmatrix}^T,$$

$$V_{n+1/2} = (v_{n+1/2,1},\ v_{n+1/2,2}, \cdots,\ v_{n+1/2,s})^T,\ U_n = (u_{n,1},\ u_{n,2},\ \cdots,\ u_{n,s})^T,$$
$$V'_{n+1/2} = (v'_{n+1/2,1},\ v'_{n+1/2,2},\ \cdots,\ v'_{n+1/2,s})^T,$$
$$U'_{n+1} = (u'_{n+1,1},\ u'_{n+1,2},\ \cdots,\ u'_{n+1,s})^T.$$

Eliminating $V_{n+1/2},\ U_n,\ U'_{n+1}$ and $V'_{n+1/2}$, we can rewrite (8) as

$$\begin{pmatrix} u_{n+1} \\ v_{n+3/2} \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}^{-1} R(\tau\omega) \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_{n+1/2} \end{pmatrix}. \tag{9}$$

For $\theta \geq 0$, $R(\theta)$ is given by

$$R(\theta) = \begin{pmatrix} 1 + r_{1,1}(\theta)\mathbf{1} & r_{1,2}(\theta)\mathbf{1} \\ r'_{1,2}(\theta)\mathbf{1}(r_{1,1}(\theta)\mathbf{1} + 1) & 1 + r'_{1,2}(\theta)\mathbf{1}r_{1,2}(\theta)\mathbf{1} + r'_{1,1}(\theta)\mathbf{1} \end{pmatrix} \tag{10}$$

with

$$r_{1,1}(\theta) = -\theta^2 d(I_s + \theta^2 AB)^{-1}A, \quad r_{1,2}(\theta) = \theta d(I_s + \theta^2 AB)^{-1},$$
$$r'_{1,1}(\theta) = -\theta^2 d'(I_s + \theta^2 A'B')^{-1}A', \quad r'_{1,2}(\theta) = -\theta d'(I_s + \theta^2 A'B')^{-1}.$$

Noticing $(\theta^2 AB)^s = O$ and $(\theta^2 A'B')^s = O$, we get

$$(I_s + \theta^2 AB)^{-1} = \sum_{i=0}^{s-1}(-\theta^2 AB)^i, \quad (I_s + \theta^2 A'B')^{-1} = \sum_{i=0}^{s-1}(-\theta^2 A'B')^i$$

with $(-\theta^2 AB)^0 = (-\theta^2 A'B')^0 = I_s$. Then we rewrite the coefficients in (10) as

$$\begin{aligned} r_{1,1}(\theta) &= d\sum_{i=0}^{s-1}(-\theta^2)^{i+1}(AB)^i A, \quad r_{1,2}(\theta) = d\sum_{i=0}^{s-1}(-\theta^2)^i\theta(AB)^i, \\ r'_{1,1}(\theta) &= d'\sum_{i=0}^{s-1}(-\theta^2)^{i+1}(A'B')^i A', \quad r'_{1,2}(\theta) = -d'\sum_{i=0}^{s-1}(-\theta^2)^i\theta(A'B')^i. \end{aligned} \tag{11}$$

Let $\lambda_\pm = \lambda_\pm(\theta)$ be the eigenvalues of (10). We know these eigenvalues are roots of

$$\begin{aligned} \lambda^2 - (2 + r_{1,1}(\theta)\mathbf{1} + r'_{1,1}(\theta)\mathbf{1} + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1})\lambda \\ +(1 + r_{1,1}(\theta)\mathbf{1})(1 + r'_{1,1}(\theta)\mathbf{1}) = 0. \end{aligned} \tag{12}$$

Under this notation, we define the stability interval of the scheme.

Definition 1. *The stability interval* S *of a staggered RK scheme* (4)-(5) *is defined by a connected closed interval of* $\{\theta; |\lambda_\pm(\theta)| \le 1,\ \theta \ge 0\}$*, which includes* 0.

The simplest example of staggered RK schemes is the (staggered) leapfrog scheme (see, e.g., [15])

$$\begin{aligned} u_{n+1} &= u_n + \tau f(t_{n+1/2}, v_{n+1/2}), \\ v_{n+3/2} &= v_{n+1/2} + \tau g(t_{n+1}, u_{n+1}). \end{aligned} \tag{13}$$

This scheme is of order 2 for ODEs. In this case, the scheme for (7) is reduced to (9) with

$$r_{1,1}(\theta)\mathbf{1} = r'_{1,1}(\theta)\mathbf{1} = 0,\ r_{1,2}(\theta)\mathbf{1} = \theta,\ r'_{1,2}(\theta)\mathbf{1} = -\theta. \tag{14}$$

Substituting (14) into (12), we get $\lambda^2 - (2-\theta^2)\lambda + 1 = 0$. Since the discriminant of $\lambda^2 - (2-\theta^2)\lambda + 1 = 0$ is $D(\theta) = (2-\theta^2)^2 - 4$, it is easy to see that $|\lambda_\pm(\theta)| \le 1$ iff $D(\theta) \le 0$. S is estimated by using the smallest positive root of $-2 = 2 - \theta^2$, i.e. $S = [0, 2]$.
RKS4 is another example of staggered RK schemes (see, [5]). This scheme is given by taking

$$A = A' = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ B = B' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ d = d' = \left(\frac{11}{12}, \frac{1}{24}, \frac{1}{24}\right). \tag{15}$$

This scheme is of order 4 for ODEs. In this case, the scheme for (7) is reduced to (9) with

$$r_{1,1}(\theta)\mathbf{1} = r'_{1,1}(\theta)\mathbf{1} = 0,\ r_{1,2}(\theta)\mathbf{1} = \theta - \frac{\theta^3}{24},\ r'_{1,2}(\theta)\mathbf{1} = -\theta + \frac{\theta^3}{24}. \tag{16}$$

Substituting (16) into (12), we get

$$\lambda^2 - \left\{2 - \left(\theta - \theta^3/24\right)^2\right\}\lambda + 1 = 0.$$

In [15], S is estimated by using the smallest positive root of $-2 = 2 - (\theta - \theta^3/24)^2$, i.e. $S = [0, 2(2^{1/3} + 2^{2/3})]$.

3. Stability of staggered RK schemes

We use (9) to estimate the stability of the staggered RK scheme. In order to prove convergence of the staggered RK scheme in the next section, we have to evaluate $||R(\theta)^n||_2$ of (10), where $||\cdot||_2$ is the Euclidean norm on $\mathbb{R}^2$ and the corresponding operator norm for 2×2 matrices. To accomplish this evaluation, we define another stability interval.
Let $\gamma_0 > 0$ ($\gamma_0 \in S$) be the smallest positive root of

$$D(\theta) = r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}\{r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1} + 4\} = 0. \tag{17}$$

By using this γ_0, we define another stability interval $S' = [0, \gamma_0)$. It is easy to see that S' is a subset in S. We prove the boundedness of $||R(\theta)^n||_2$ by using following hypotheses for the staggered RK scheme (4)-(5):

(H1) For $\theta \in S'$, $0 \le -r'_{1,2}(\theta)\mathbf{1} \le r_{1,2}(\theta)\mathbf{1} \le -\gamma_0 r'_{1,2}(\theta)\mathbf{1}$.

(H2) For $\theta \in S'$, $D(\theta) \le 0$.

(H3) The polynomials $r_{1,1}(\theta)\mathbf{1}$ and $r'_{1,1}(\theta)\mathbf{1}$ are 0.

(H4) The following order condition holds: $d\mathbf{1} = d'\mathbf{1} = 1$.

The leapfrog scheme (13) and RKS4 (15) satisfy these hypotheses. Substituting (14) into (17), we can take $\gamma_0 = 2$ and $S' = [0, 2)$ for the leapfrog scheme. By (14), the leapfrog scheme satisfies (H1)-(H3). (H4) is checked by using (13). Similarly, we can take $\gamma_0 = 2\sqrt{6}$ and $S' = \left[0, 2\sqrt{6}\right)$ for RKS4, by substituting (14) and (16) into (17). By (16), RKS4 satisfies (H1)-(H3). By (15), (H4) holds.

Theorem 3.1. *Let $\gamma_\varepsilon > 0$ be $\gamma_\varepsilon < \gamma_0$. Assume that the coefficients $a_{i,j}$, $a'_{i,j}$, $b_{i,j}$, $b'_{i,j}$, c_i, c'_i, d_i, d'_i, e_i, e'_i in* (4)-(5) *satisfy (H1)-(H4). Then, there is a positive constant C such that*

$$||R(\theta)^n||_2 \leq C \tag{18}$$

holds for any $0 \leq \theta \leq \gamma_\varepsilon$ and $n \in \mathbb{N}$. Here $R(\theta)$ is the matrix of (10).

Proof. By (H3), we can rewrite

$$R(\theta) = \begin{pmatrix} 1 & r_{1,2}(\theta)\mathbf{1} \\ r'_{1,2}(\theta)\mathbf{1} & 1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1} \end{pmatrix}. \tag{19}$$

If $\theta = 0$, $R(\theta)$ is the identity matrix. Then (18) holds for $C = 1$. Let $\theta > 0$. We can diagonalize (19) as

$$R(\theta) = Q(\theta) \begin{pmatrix} \lambda_+(\theta) & 0 \\ 0 & \lambda_-(\theta) \end{pmatrix} Q(\theta)^{-1}. \tag{20}$$

Here

$$\lambda_\pm(\theta) = \lambda_\pm = \frac{2 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1} \pm \sqrt{D(\theta)}}{2}, \tag{21}$$

$$Q(\theta) = \frac{1}{r'_{1,2}(\theta)\mathbf{1}} \begin{pmatrix} \lambda_+ - (1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}) & \lambda_- - (1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}) \\ r'_{1,2}(\theta)\mathbf{1} & r'_{1,2}(\theta)\mathbf{1} \end{pmatrix},$$

$$Q(\theta)^{-1} = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} r'_{1,2}(\theta)\mathbf{1} & -\lambda_- + (1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}) \\ -r'_{1,2}(\theta)\mathbf{1} & \lambda_+ - (1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}) \end{pmatrix}.$$

Since $\theta \in S$, we have $|\lambda_\pm| \leq 1$. By (H2), the adjoint matrices of $Q(\theta)$ and $Q(\theta)^{-1}$ are

$$Q(\theta)^* = \frac{1}{r'_{1,2}(\theta)\mathbf{1}} \begin{pmatrix} \lambda_- - (1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}) & r'_{1,2}(\theta)\mathbf{1} \\ \lambda_+ - (1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}) & r'_{1,2}(\theta)\mathbf{1} \end{pmatrix},$$

$$(Q(\theta)^{-1})^* = \frac{1}{\lambda_- - \lambda_+} \begin{pmatrix} r'_{1,2}(\theta)\mathbf{1} & -r'_{1,2}(\theta)\mathbf{1} \\ -\lambda_+ + (1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}) & \lambda_- - (1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}) \end{pmatrix}.$$

Putting

$$\begin{aligned} a(\theta) &= \{\lambda_- - (1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1})\}\{\lambda_+ - (1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1})\}, \\ b_\pm(\theta) &= \{\lambda_\pm - (1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1})\}^2 + (r'_{1,2}(\theta)\mathbf{1})^2, \\ c(\theta) &= -r'_{1,2}(\theta)\mathbf{1}\{\lambda_+ + \lambda_- - 2(1 + r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1})\}, \end{aligned}$$

we have

$$Q(\theta)^*Q(\theta) = \frac{1}{(r'_{1,2}(\theta)\mathbf{1})^2}\begin{pmatrix} a(\theta)+(r'_{1,2}(\theta)\mathbf{1})^2 & b_-(\theta) \\ b_+(\theta) & a(\theta)+(r'_{1,2}(\theta)\mathbf{1})^2 \end{pmatrix},$$

$$(Q(\theta)^{-1})^*(Q(\theta)^{-1}) = \frac{-1}{\{\lambda_- - \lambda_+\}^2}\begin{pmatrix} 2r'_{1,2}(\theta)^2 & c(\theta) \\ c(\theta) & 2a(\theta) \end{pmatrix}.$$

Then, the eigenvalues of $Q(\theta)^*Q(\theta)$ and $(Q(\theta)^{-1})^*(Q(\theta)^{-1})$ are

$$\frac{a(\theta)+(r'_{1,2}(\theta)\mathbf{1})^2 \pm \sqrt{b_-(\theta)b_+(\theta)}}{(r'_{1,2}(\theta)\mathbf{1})^2},$$

$$\frac{a(\theta)+(r'_{1,2}(\theta)\mathbf{1})^2 \pm \sqrt{(a(\theta)+(r'_{1,2}(\theta)\mathbf{1})^2)^2 - 4a(\theta)(r'_{1,2}(\theta)\mathbf{1})^2 + c(\theta)^2}}{-\{\lambda_- - \lambda_+\}^2},$$

respectively. Putting

$$\begin{aligned} \alpha(\theta) &= -r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1} + (r'_{1,2}(\theta)\mathbf{1})^2, \\ \beta(\theta) &= r'_{1,2}(\theta)\mathbf{1}(\lambda_+ - \lambda_-)i, \end{aligned} \tag{22}$$

these eigenvalues are rewritten as

$$\frac{\alpha(\theta) \pm \sqrt{\alpha(\theta)^2 - \beta(\theta)^2}}{(r_{2,1}(\theta)\mathbf{1})^2}, \quad \frac{(r_{2,1}(\theta)\mathbf{1})^2\left\{\alpha(\theta) \pm \sqrt{\alpha(\theta)^2 - \beta(\theta)^2}\right\}}{\beta(\theta)^2},$$

respectively. Then, by (20), we have

$$||R(\theta)^n||_2 \le ||Q(\theta)||_2 \left|\left|Q(\theta)^{-1}\right|\right|_2 = \left|\frac{\alpha(\theta) + \sqrt{\alpha(\theta)^2 - \beta(\theta)^2}}{\beta(\theta)}\right| \le 2\left|\frac{\alpha(\theta)}{\beta(\theta)}\right| + 1. \tag{23}$$

Substituting (21) into (22) and using (H1), we have

$$\begin{aligned} \left|\frac{\alpha(\theta)}{\beta(\theta)}\right| &= \frac{|r_{1,2}(\theta)\mathbf{1} - r'_{1,2}(\theta)\mathbf{1}|}{\sqrt{-r'_{1,2}(\theta)\mathbf{1}r_{1,2}(\theta)\mathbf{1}(r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}+4)}} \\ &\le \frac{(1+\gamma_0)r'_{1,2}(\theta)\mathbf{1}}{r'_{1,2}(\theta)\mathbf{1}\sqrt{r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}+4}} \end{aligned}$$

for any $\theta \in [0, \gamma_\varepsilon]$. By (H1) and (H2), we get $-4 \le r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1} \le 0$. As $r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}$ is a polynomial of θ, there exits a minimum value of $r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}+4$ in $[0, \gamma_\varepsilon]$. Let γ_1 be the value of θ that gives the minimum value of $r_{1,2}(\theta)\mathbf{1}r'_{1,2}(\theta)\mathbf{1}+4$. We get

$$\left|\frac{\alpha(\theta)}{\beta(\theta)}\right| \le \frac{1+\gamma_0}{\sqrt{r_{1,2}(\gamma_1)\mathbf{1}r_{2,1}(\gamma_1)\mathbf{1}+4}}.$$

Then, this, together with (23), gives (18) with $C = \dfrac{2(1+\gamma_0)}{\sqrt{r_{1,2}(\gamma_1)\mathbf{1}r_{2,1}(\gamma_1)\mathbf{1}+4}} + 1$. □

4. Convergence of fully discrete schemes

We assume the following hypotheses for L_h:

L_h is a negative definite symmetric matrix.

There exits $h_0 > 0$ and $C_3 > 0$ such that any eigenvalues of L_h is less than $-C_3$ for any $h < h_0$.

Form these hypotheses, we can take a positive definite symmetric matrix W_h satisfying $-DL_h = W_h^2$; Any eigenvalues of W_h^{-1} is less than $1/\sqrt{DC_3}$ for any $h < h_0$.
Using W_h, we can rewrite (2) as

$$\frac{du_h(t)}{dt} = v_h(t), \quad \frac{dv_h(t)}{dt} = -W_h^2 u_h(t) + \varphi_h(t) + g_h(t, u_h(t)). \tag{24}$$

In this paper, $||\cdot||_{W_h}$ denotes a discrete energy norm (see, e.g., [1], [2]), given by

$$||(u_h, v_h)^T||_{W_h}^2 = ||W_h u_h||^2 + ||v_h||^2 \quad \text{for any } u_h, v_h \in \boldsymbol{V}_h, \tag{25}$$

where $||\cdot||$ denotes the discrete version of the L_2-norm in $\boldsymbol{V}_h$, given by

$$||u_h||^2 = h \sum_{x \in \Omega_h} \{(u_h)_x\}^2$$

and the corresponding operator norm for $m \times m$ matrices with $m = \dim \boldsymbol{V}_h$.
We define the spatial truncation error $\alpha_h(t)$ by

$$\alpha_h(t) = \boldsymbol{v}'_h(t) + W_h^2 \boldsymbol{u}_h(t) - \varphi_h(t) - g_h(t, \boldsymbol{u}_h(t)), \tag{26}$$

where $\boldsymbol{u}_h(t),\ \boldsymbol{v}_h(t)$ are $\boldsymbol{V}_h$-valued functions obtained by restricting the variable x of the exact solutions $u,\ v$ onto Ω_h.
By applying (4)-(5) to (24), we obtain the following scheme for the problem (1):

$$\begin{aligned}
\boldsymbol{V}_{n+1/2} &= \mathbf{1}'\boldsymbol{v}_{n+1/2} + \tau\boldsymbol{A}\{-\boldsymbol{W}_h^2\boldsymbol{U}_n + \boldsymbol{\varphi}_h(t_n) + \boldsymbol{g}_n\},\\
\boldsymbol{U}_n &= \mathbf{1}'\boldsymbol{u}_n + \tau\boldsymbol{B}\boldsymbol{V}_{n+1/2},\\
\boldsymbol{u}_{n+1} &= \boldsymbol{u}_n + \tau\boldsymbol{d}\boldsymbol{V}_{n+1/2},\\
\boldsymbol{U}'_{n+1} &= \mathbf{1}'\boldsymbol{u}_{n+1} + \tau\boldsymbol{A}'\boldsymbol{V}'_{n+1/2},\\
\boldsymbol{V}'_{n+1/2} &= \mathbf{1}'\boldsymbol{v}_{n+1/2} + \tau\boldsymbol{B}'\{-\boldsymbol{W}_h^2\boldsymbol{U}'_{n+1} + \boldsymbol{\varphi}_h(t_{n+1}) + \boldsymbol{g}_{n+1}\},\\
\boldsymbol{v}_{n+3/2} &= \boldsymbol{v}_{n+1/2} + \tau\boldsymbol{d}'\{-\boldsymbol{W}_h^2\boldsymbol{U}'_{n+1} + \boldsymbol{\varphi}_h(t_{n+1}) + \boldsymbol{g}_{n+1}\}.
\end{aligned} \tag{27}$$

Here $\mathbf{1}'$ denotes $\mathbf{1}\otimes I_m$ for $\mathbf{1} = (1,\cdots,1)^T \in \mathbb{R}^s$,

$$\begin{aligned}
&\boldsymbol{A} = A\otimes I_m,\ \boldsymbol{B} = B\otimes I_m,\ \boldsymbol{d} = d\otimes I_m,\ \boldsymbol{A}' = A'\otimes I_m,\ \boldsymbol{B}' = B'\otimes I_m,\\
&\boldsymbol{V}_{n+1/2} = (\boldsymbol{v}^T_{n+1/2,1}, \boldsymbol{v}^T_{n+1/2,2}, \cdots, \boldsymbol{v}^T_{n+1/2,s})^T,\ \boldsymbol{U}_n = (\boldsymbol{u}^T_{n,1}, \boldsymbol{u}^T_{n,2}, \cdots, \boldsymbol{u}^T_{n,s})^T,\\
&\boldsymbol{V}'_{n+1/2} = (\boldsymbol{v}'^T_{n+1/2,1}, \boldsymbol{v}'^T_{n+1/2,2}, \cdots, \boldsymbol{v}'^T_{n+1/2,s})^T,\\
&\boldsymbol{U}'_{n+1} = (\boldsymbol{u}'^T_{n+1,1}, \boldsymbol{u}'^T_{n+1,2}, \cdots, \boldsymbol{u}'^T_{n+1,s})^T,\\
&\boldsymbol{\varphi}_h(t_n) = (\varphi_h(t_{n,1})^T, \varphi_h(t_{n,2})^T, \cdots, \varphi_h(t_{n,s})^T)^T,\ \boldsymbol{d}' = d'\otimes I_m,\\
&\boldsymbol{g}_n = (g_h(t_{n,1}, \boldsymbol{u}_{n,1})^T, g_h(t_{n,2}, \boldsymbol{u}_{n,2})^T, \cdots, g_h(t_{n,s}, \boldsymbol{u}_{n,s})^T)^T,\ \boldsymbol{W}_h = I_s\otimes W_h
\end{aligned}$$

with $\otimes$ standing for the Kronecker product (see, e.g., [4]), $\boldsymbol{u}_{n,i}$, $\boldsymbol{v}_{n+1/2,i}$, $\boldsymbol{u}'_{n+1,i}$ and $\boldsymbol{v}'_{n+1/2,i}$ are intermediate variables, $t_{n,j} := t_n + c_j\tau$, $t_{n+1,j} := t_{n+1} + c'_j\tau$, $\boldsymbol{u}_n$ and $\boldsymbol{v}_{n+1/2}$ are approximate values of $\boldsymbol{u}_h(t_n)$ and $\boldsymbol{v}_h(t_{n+1/2})$, respectively.
For some s-dimensional vector $\boldsymbol{a} = (a_1,\cdots,a_s)^T$, we define $\boldsymbol{a}^i = (a_1^i,\cdots,a_s^i)^T$.
In addition to the (H1)-(H4), we assume the following hypothesis for the staggered RK scheme (4)-(5):

(H5) The following order conditions hold:

$$\begin{aligned}
&(A\mathbf{1})^2 + A\mathbf{1} = 2AB\mathbf{1},\ (B\mathbf{1})^2 - B\mathbf{1} = 2BA\mathbf{1},\\
&(A'\mathbf{1})^2 + A'\mathbf{1} = 2A'B'\mathbf{1},\ (B'\mathbf{1})^2 - B'\mathbf{1} = 2B'A'\mathbf{1},\\
&dA\mathbf{1} = d'A'\mathbf{1} = 0.
\end{aligned}$$

The leapfrog scheme and RKS4 satisfy (H5), which is checked by (13) and (15).
We assume the following condition which gives the restriction for τ and h.

(H6) $\tau\rho(W_h) \in S'$. Here $\rho(W_h)$ is a spectral radius of W_h.

We put the coefficients of (4)-(5) as

$$\zeta = \frac{4(A\mathbf{1})^3 + 6(A\mathbf{1})^2 + 3(A\mathbf{1})}{24} - \frac{A(B\mathbf{1})^2}{2},$$
$$\eta = \frac{4(B\mathbf{1})^3 - 6(B\mathbf{1})^2 + 3(B\mathbf{1})}{24} - \frac{B(A\mathbf{1})^2}{2},$$
$$\zeta' = \frac{4(A'\mathbf{1})^3 + 6(A'\mathbf{1})^2 + 3(A'\mathbf{1})}{24} - \frac{A'(B'\mathbf{1})^2}{2},$$
$$\eta' = \frac{4(B'\mathbf{1})^3 - 6(B'\mathbf{1})^2 + 3(B'\mathbf{1})}{24} - \frac{B'(A'\mathbf{1})^2}{2}.$$

Moreover, we assume the following condition for the problem (1):
The exact solution $u(t,x)$ is of class C^4 with respect to t, $g(t,x,u)$ is of class C^3 with respect to t, u and (each component of) the derivative $\partial g/\partial u$ is bounded for $(t,x,u) \in [0,T] \times \Omega \times \mathbb{R}$.
For simplicity, we consider a step size of the form $\tau = T/N$ with positive integer N. Then, we have the following theorem.

Theorem 4.1. *Assume that the coefficients $a_{i,j}$, $a'_{i,j}$, $b_{i,j}$, $b'_{i,j}$, c_i, c'_i, d_i, d'_i, e_i, e'_i in* (4)-(5) *satisfy (H1)-(H5) and τ satisfies (H6). Then, there is a positive constant C_1 such that*

$$\left\|\left(\boldsymbol{u}_n - \boldsymbol{u}_h(t_n), \boldsymbol{v}_{n+1/2} - \boldsymbol{v}_h(t_{n+1/2})\right)^T\right\|_{W_h} \leq C_1 \left(\tau^2 + \max_{0\leq t\leq T} ||\alpha_h(t)||\right) \quad (28)$$

holds.

Proof. Put

$$\boldsymbol{V}_h(t_{n+1/2}) = (\boldsymbol{v}_h(t_{n+1/2,1})^T, \boldsymbol{v}_h(t_{n+1/2,2})^T, \cdots, \boldsymbol{v}_h(t_{n+1/2,s})^T)^T,$$
$$\boldsymbol{U}_h(t_n) = (\boldsymbol{u}_h(t_{n,1})^T, \boldsymbol{u}_h(t_{n,2})^T, \cdots, \boldsymbol{u}_h(t_{n,s})^T)^T,$$
$$\boldsymbol{V}_h(t'_{n+1/2}) = (\boldsymbol{v}_h(t'_{n+1/2,1})^T, \boldsymbol{v}_h(t'_{n+1/2,2})^T, \cdots, \boldsymbol{v}_h(t'_{n+1/2,s})^T)^T,$$
$$\boldsymbol{g}_h(t_n) = (g_h(t_{n,1}, \boldsymbol{u}_h)^T, g_h(t_{n,2}, \boldsymbol{u}_h)^T, \cdots, g_h(t_{n,s}, \boldsymbol{u}_h)^T)^T,$$

where $t_{n+1/2,j} := t_{n+1/2} + e_j\tau$, $t'_{n+1/2,j} := t_{n+1/2} + e'_j\tau$, $j = 1, \cdots, s$. Replacing $\boldsymbol{U}_n$, $\boldsymbol{U}'_{n+1}$, $\boldsymbol{V}_{n+1/2}$, $\boldsymbol{V}'_{n+1/2}$, $\boldsymbol{u}_n$ and $\boldsymbol{v}_{n+1/2}$ in the scheme (27) with

$\boldsymbol{U}_h(t_n)$, $\boldsymbol{U}_h(t_{n+1})$, $\boldsymbol{V}_h(t_{n+1/2})$, $\boldsymbol{V}_h(t'_{n+1/2})$, $\boldsymbol{u}_h(t_n)$ and $\boldsymbol{v}_h(t_{n+1/2})$, we obtain the recurrence relation

$$
\begin{aligned}
\boldsymbol{V}_h(t_{n+1/2}) &= \boldsymbol{1}'\boldsymbol{v}_h(t_{n+1/2}) + \tau\boldsymbol{A}\{-\boldsymbol{W}_h^2\boldsymbol{U}_h(t_n) + \boldsymbol{\varphi}_h(t_n) + \boldsymbol{g}_h(t_n)\} + \boldsymbol{r}_{n+1/2},\\
\boldsymbol{U}_h(t_n) &= \boldsymbol{1}'\boldsymbol{u}_h(t_n) + \tau\boldsymbol{B}\boldsymbol{V}_h(t_{n+1/2}) + \boldsymbol{r}_n,\\
\boldsymbol{u}_h(t_{n+1}) &= \boldsymbol{u}_h(t_n) + \tau\boldsymbol{d}\boldsymbol{V}_h(t_{n+1/2}) + \rho_n,\\
\boldsymbol{U}_h(t_{n+1}) &= \boldsymbol{1}'\boldsymbol{u}_h(t_{n+1}) + \tau\boldsymbol{A}'\boldsymbol{V}_h(t'_{n+1/2}) + \boldsymbol{r}_{n+1},\\
\boldsymbol{V}_h(t'_{n+1/2}) &= \boldsymbol{1}'\boldsymbol{v}_h(t_{n+1/2}) + \tau\boldsymbol{B}'\{-\boldsymbol{W}_h^2\boldsymbol{U}_h(t_{n+1}) + \boldsymbol{\varphi}_h(t_{n+1}) + \boldsymbol{g}_h(t_{n+1})\} + \boldsymbol{r}'_{n+1/2},\\
\boldsymbol{v}_h(t_{n+3/2}) &= \boldsymbol{v}_h(t_{n+1/2}) + \tau\boldsymbol{d}'\{-\boldsymbol{W}_h^2\boldsymbol{U}_h(t_{n+1}) + \boldsymbol{\varphi}_h(t_{n+1}) + \boldsymbol{g}_h(t_{n+1})\} + \rho_{n+1/2}
\end{aligned}
\tag{29}
$$

with the residuals

$$\boldsymbol{r}_n = (r_{n,1}^T, r_{n,2}^T, \cdots, r_{n,s}^T)^T,\ \boldsymbol{r}'_{n+1/2} = (r'^T_{n+1/2,1}, r'^T_{n+1/2,2}, \cdots, r'^T_{n+1/2,s})^T,$$

ρ_n and $\rho_{n+1/2}$. By (6), (26), (H4) and (H5), these residuals are expanded as

$$
\begin{aligned}
\boldsymbol{r}_{n+1/2} &= \tau^3\boldsymbol{\zeta}\boldsymbol{v}_h^{(3)}(t_{n+1/2}) + \tau\boldsymbol{A}\boldsymbol{\alpha}_h(t_n) + \mathrm{O}(\tau^4),\\
\boldsymbol{r}_n &= \tau^3\boldsymbol{\eta}\boldsymbol{u}_h^{(3)}(t_n) + \mathrm{O}(\tau^4),\\
\rho_n &= \frac{\tau^3}{2}\left(\frac{1}{12} - d(A\boldsymbol{1})^2\right)\boldsymbol{u}_h^{(3)}(t_n) + \mathrm{O}(\tau^4),\\
\boldsymbol{r}_{n+1} &= \tau^3\boldsymbol{\zeta}'\boldsymbol{u}_h^{(3)}(t_{n+1}) + \mathrm{O}(\tau^4),\\
\boldsymbol{r}'_{n+1/2} &= \tau^3\boldsymbol{\eta}'\boldsymbol{v}_h^{(3)}(t_{n+1/2}) + \tau\boldsymbol{B}'\boldsymbol{\alpha}_h(t_{n+1}) + \mathrm{O}(\tau^4),\\
\rho_{n+1/2} &= \frac{\tau^3}{2}\left(\frac{1}{12} - d'(A'\boldsymbol{1})^2\right)\boldsymbol{v}_h^{(3)}(t_{n+1/2}) + \tau\boldsymbol{d}'\boldsymbol{\alpha}_h(t_{n+1}) + \mathrm{O}(\tau^4).
\end{aligned}
\tag{30}
$$

Here

$$
\begin{aligned}
&\boldsymbol{\alpha}_h(t_n) = (\alpha_h(t_{n,1})^T, \alpha_h(t_{n,2})^T, \cdots, \alpha_h(t_{n,s})^T)^T,\\
&\boldsymbol{\zeta} = \zeta\otimes I_m,\ \boldsymbol{\eta} = \eta\otimes I_m,\ \boldsymbol{\zeta}' = \zeta'\otimes I_m,\ \boldsymbol{\eta}' = \eta'\otimes I_m,
\end{aligned}
$$

$O(\tau^4)$ denotes a term whose component for each $x \in \Omega_h$ is of $O(\tau^4)$. Subtracting (27) from (29), we obtain

$$\begin{aligned}
\boldsymbol{\delta}_{n+1/2} &= \mathbf{1}'\varepsilon_{n+1/2} - \tau \boldsymbol{A}(\boldsymbol{W}_h^2 \boldsymbol{\delta}_n - \boldsymbol{g}_h(t_n) + \boldsymbol{g}_n) + \boldsymbol{r}_{n+1/2}, \\
\boldsymbol{\delta}_n &= \mathbf{1}'\varepsilon_n + \tau \boldsymbol{B}\boldsymbol{\delta}_{n+1/2} + \boldsymbol{r}_n, \\
\varepsilon_{n+1} &= \varepsilon_n + \tau \boldsymbol{d}\boldsymbol{\delta}_{n+1/2} + \rho_n, \\
\boldsymbol{\delta}'_{n+1} &= \mathbf{1}'\varepsilon_{n+1} + \tau \boldsymbol{A}'\boldsymbol{\delta}'_{n+1/2} + \boldsymbol{r}_{n+1}, \\
\boldsymbol{\delta}'_{n+1/2} &= \mathbf{1}'\varepsilon_{n+1/2} - \tau \boldsymbol{B}'(\boldsymbol{W}_h^2 \boldsymbol{\delta}'_{n+1} - \boldsymbol{g}_h(t_{n+1}) + \boldsymbol{g}_{n+1}) + \boldsymbol{r}'_{n+1/2}, \\
\varepsilon_{n+3/2} &= \varepsilon_{n+1/2} - \tau \boldsymbol{d}'(\boldsymbol{W}_h^2 \boldsymbol{\delta}'_{n+1} - \boldsymbol{g}_h(t_{n+1}) + \boldsymbol{g}_{n+1}) + \rho_{n+1/2}.
\end{aligned}$$

Here

$$\begin{aligned}
&\boldsymbol{\delta}_{n+1/2} = \boldsymbol{V}_h(t_{n+1/2}) - \boldsymbol{V}_{n+1/2},\ \boldsymbol{\delta}_n = \boldsymbol{U}_h(t_n) - \boldsymbol{U}_n, \\
&\boldsymbol{\delta}'_{n+1} = \boldsymbol{U}_h(t_{n+1}) - \boldsymbol{U}'_{n+1},\ \boldsymbol{\delta}'_{n+1/2} = \boldsymbol{V}_h(t'_{n+1/2}) - \boldsymbol{V}'_{n+1/2}
\end{aligned}$$

for the errors

$$\varepsilon_n = \boldsymbol{u}_h(t_n) - \boldsymbol{u}_n,\ \varepsilon_{n+1/2} = \boldsymbol{v}_h(t_{n+1/2}) - \boldsymbol{v}_{n+1/2}.$$

Let $\boldsymbol{J}_n$ be $\boldsymbol{J}_n = \mathrm{diag}(J_{n,1}, J_{n,2}, \cdots, J_{n,s})$ and $J_{n,i}$ be a function from Ω_h to $\mathbb{R}$ whose value for $x \in \Omega_h$ is

$$J_{n,i}(x) = \int_0^1 \frac{\partial g}{\partial u}(t_{n,i}, x, (1-\theta)\boldsymbol{u}_{n,i}(x) + \theta \boldsymbol{u}_h(t_{n,i}, x))d\theta.$$

By the assumption that $\partial g/\partial u$ is bounded, there is a constant γ_3 such that

$$||J_{n,i}v|| \le \gamma_3 ||v|| \quad \text{for any } v \in \boldsymbol{V}_h, \tag{31}$$

where the multiplication $J_{n,i}v$ is component-wise for $x \in \Omega_h$. Then we obtain

$$\begin{aligned}
\boldsymbol{\delta}_{n+1/2} &= \mathbf{1}'\varepsilon_{n+1/2} - \tau \boldsymbol{A}(\boldsymbol{W}_h^2 - \boldsymbol{J}_n)\boldsymbol{\delta}_n + \boldsymbol{r}_{n+1/2}, \\
\boldsymbol{\delta}_n &= \mathbf{1}'\varepsilon_n + \tau \boldsymbol{B}\boldsymbol{\delta}_{n+1/2} + \boldsymbol{r}_n, \\
\varepsilon_{n+1} &= \varepsilon_n + \tau \boldsymbol{d}\boldsymbol{\delta}_{n+1/2} + \rho_n, \\
\boldsymbol{\delta}'_{n+1} &= \mathbf{1}'\varepsilon_{n+1} + \tau \boldsymbol{A}'\boldsymbol{\delta}'_{n+1/2} + \boldsymbol{r}_{n+1}, \\
\boldsymbol{\delta}'_{n+1/2} &= \mathbf{1}'\varepsilon_{n+1/2} - \tau \boldsymbol{B}'(\boldsymbol{W}_h^2 - \boldsymbol{J}_{n+1})\boldsymbol{\delta}'_{n+1} + \boldsymbol{r}'_{n+1/2}, \\
\varepsilon_{n+3/2} &= \varepsilon_{n+1/2} - \tau \boldsymbol{d}'(\boldsymbol{W}_h^2 - \boldsymbol{J}_{n+1})\boldsymbol{\delta}'_{n+1} + \rho_{n+1/2}.
\end{aligned}$$

Eliminating $\boldsymbol{\delta}_n$, $\boldsymbol{\delta}_{n+1/2}$, $\boldsymbol{\delta}'_{n+1/2}$ and $\boldsymbol{\delta}_{n+1}$, we have

$$\begin{pmatrix} W_h\varepsilon_{n+1} \\ \varepsilon_{n+3/2} \end{pmatrix} = \boldsymbol{R}_n \begin{pmatrix} W_h\varepsilon_n \\ \varepsilon_{n+1/2} \end{pmatrix} + \boldsymbol{M}_n \begin{pmatrix} W_h\xi_n \\ \xi_{n+1/2} \end{pmatrix}. \tag{32}$$

Here

$$\begin{aligned}
&\boldsymbol{R}_n = \begin{pmatrix} I_m + R_{1,1}\mathbf{1}' & R_{1,2}\mathbf{1}' \\ R'_{1,2}\mathbf{1}'R_{1,1}\mathbf{1}' + R'_{1,2}\mathbf{1}' & I_m + R'_{1,2}\mathbf{1}'R_{1,2}\mathbf{1}' + R'_{1,1}\mathbf{1}' \end{pmatrix},\ \boldsymbol{M}_n = \begin{pmatrix} I_m & O \\ R'_{1,2}\mathbf{1}' & I_m \end{pmatrix}, \\
&R_{1,1} = -\tau^2\boldsymbol{d}(\boldsymbol{I} + \tau^2\boldsymbol{A}(\boldsymbol{W}_h^2 - \boldsymbol{J}_n)\boldsymbol{B})^{-1}\boldsymbol{A}(\boldsymbol{W}_h^2 - \boldsymbol{J}_n), \\
&R_{1,2} = \tau\boldsymbol{d}(\boldsymbol{I} + \tau^2\boldsymbol{A}(\boldsymbol{W}_h^2 - \boldsymbol{J}_n)\boldsymbol{B})^{-1}\boldsymbol{W}_h, \\
&R'_{1,1} = -\tau^2\boldsymbol{d}'(\boldsymbol{W}_h^2 - \boldsymbol{J}_{n+1})(\boldsymbol{I} + \tau^2\boldsymbol{A}'\boldsymbol{B}'(\boldsymbol{W}_h^2 - \boldsymbol{J}_{n+1}))^{-1}\boldsymbol{A}', \\
&R'_{1,2} = -\tau\boldsymbol{d}'(\boldsymbol{W}_h^2 - \boldsymbol{J}_{n+1})(\boldsymbol{I} + \tau^2\boldsymbol{A}'\boldsymbol{B}'(\boldsymbol{W}_h^2 - \boldsymbol{J}_{n+1}))^{-1}\boldsymbol{W}_h^{-1}, \\
&W_h\xi_n = R_{1,1}\boldsymbol{W}_h\boldsymbol{r}_n + R_{1,2}\boldsymbol{r}_{n+1/2} + W_h\rho_n, \\
&\xi_{n+1/2} = R'_{1,2}\boldsymbol{W}_h\boldsymbol{r}_{n+1} + R'_{1,1}\boldsymbol{r}'_{n+1/2} + \rho_{n+1/2}
\end{aligned} \tag{33}$$

with $\boldsymbol{I} = I_s \otimes I_m$.
In order to prove the convergence, we introduce new variables following [6] and [15]. As in the proof of Lemma II.2.3 in [6] and 5.3 in [15], we put

$$\begin{aligned}
\begin{pmatrix} W_h\nu_n \\ \nu_{n+1/2} \end{pmatrix} &= (R(\tau W_h) - I_{2m})^{-1}M(\tau W_h)\begin{pmatrix} W_h\psi_n \\ \psi_{n+1/2} \end{pmatrix} \\
&= \begin{pmatrix} [\boldsymbol{d}'(\boldsymbol{I} + \tau^2\boldsymbol{A}'\boldsymbol{B}'\boldsymbol{W}_h^2)^{-1}\mathbf{1}']^{-1} W_h^{-1}\tau^{-1}\psi_{n+1/2} \\ [\boldsymbol{d}(\boldsymbol{I} + \tau^2\boldsymbol{A}\boldsymbol{W}_h^2\boldsymbol{B})^{-1}\mathbf{1}']^{-1} W_h^{-1}\tau^{-1}W_h\psi_n \end{pmatrix},
\end{aligned} \tag{34}$$

$$\begin{aligned}
\begin{pmatrix} W_h\hat{\varepsilon}_n \\ \hat{\varepsilon}_{n+1/2} \end{pmatrix} &= \begin{pmatrix} W_h\varepsilon_n \\ \varepsilon_{n+1/2} \end{pmatrix} + \begin{pmatrix} W_h\nu_n \\ \nu_{n+1/2} \end{pmatrix}, \\
\begin{pmatrix} W_h\hat{\xi}_n \\ \hat{\xi}_{n+1/2} \end{pmatrix} &= \tau M(\tau W_h)\begin{pmatrix} W_h\bar{\xi}_n \\ \bar{\xi}_{n+1/2} \end{pmatrix} - \tau\bar{\boldsymbol{R}}_n\begin{pmatrix} W_h\nu_n \\ \nu_{n+1/2} \end{pmatrix} + \begin{pmatrix} W_h(\nu_{n+1} - \nu_n) \\ \nu_{n+3/2} - \nu_{n+1/2} \end{pmatrix}
\end{aligned} \tag{35}$$

and rewrite (32) as

$$\begin{pmatrix} W_h\hat{\varepsilon}_{n+1} \\ \hat{\varepsilon}_{n+3/2} \end{pmatrix} = \boldsymbol{R}_n\begin{pmatrix} W_h\hat{\varepsilon}_n \\ \hat{\varepsilon}_{n+1/2} \end{pmatrix} + \begin{pmatrix} W_h\hat{\xi}_n \\ \hat{\xi}_{n+1/2} \end{pmatrix}. \tag{36}$$

Here

$$
\begin{aligned}
&M(\tau W_h) = \begin{pmatrix} I_m & O \\ r'_{1,2}(\tau W_h)\mathbf{1}' & I_m \end{pmatrix}, \\
&W_h\psi_n = r_{1,1}(\tau W_h)\boldsymbol{W}_h\boldsymbol{r}_n + r_{1,2}(\tau W_h)\boldsymbol{r}_{n+1/2} + W_h\rho_n, \\
&\psi_{n+1/2} = r'_{1,2}(\tau W_h)\boldsymbol{W}_h\boldsymbol{r}_{n+1} + r'_{1,1}(\tau W_h)\boldsymbol{r}'_{n+1/2} + \rho_{n+1/2},
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
&W_h\bar{\xi}_n = \bar{R}_{1,1}\boldsymbol{W}_h\boldsymbol{r}_n + \bar{R}_{1,2}\boldsymbol{r}_{n+1/2}, \\
&\bar{\xi}_{n+1/2} = \bar{R}'_{1,2}\mathbf{1}'W_h\xi_n + \bar{R}'_{1,2}\boldsymbol{W}_h\boldsymbol{r}_{n+1} + \bar{R}'_{1,1}\boldsymbol{r}'_{n+1/2}.
\end{aligned} \tag{38}
$$

$\bar{\boldsymbol{R}}_n$ is defined as $\tau\bar{\boldsymbol{R}}_n = \boldsymbol{R}_n - R(\tau W_h)$, given by

$$
\bar{\boldsymbol{R}}_n = \begin{pmatrix} \bar{R}_{1,1}\mathbf{1}' & \bar{R}_{1,2}\mathbf{1}' \\ R'_{1,2}\mathbf{1}'\bar{R}_{1,1}\mathbf{1}' + \bar{R}'_{1,2}\mathbf{1}' & R'_{1,2}\mathbf{1}'\bar{R}_{1,2}\mathbf{1}' + \bar{R}'_{1,2}\mathbf{1}'r_{1,2}(\tau W_h)\mathbf{1}' + \bar{R}'_{1,1}\mathbf{1}' \end{pmatrix}.
$$

Since $\boldsymbol{A}\boldsymbol{W}_h^2\boldsymbol{B} = \boldsymbol{W}_h^2\boldsymbol{A}\boldsymbol{B},\ \boldsymbol{A}'\boldsymbol{B}'\boldsymbol{W}_h^2 = \boldsymbol{W}_h^2\boldsymbol{A}'\boldsymbol{B}',\ \bar{R}_{1,i},\ \bar{R}'_{1,i},\ i = 1,2$ are written as

$$
\begin{aligned}
\bar{R}_{1,1} &= -\tau\boldsymbol{d}\sum_{i=0}^{s-1}(-1)^i\left\{(\tau^2\boldsymbol{W}_h^2\boldsymbol{A}\boldsymbol{B} - \tau^2\boldsymbol{A}\boldsymbol{J}_n\boldsymbol{B})^i - (\tau^2\boldsymbol{W}_h^2\boldsymbol{A}\boldsymbol{B})^i\right\}\boldsymbol{A}\boldsymbol{W}_h^2 \\
&\quad + \tau\boldsymbol{d}\sum_{i=0}^{s-1}(\tau^2\boldsymbol{A}(\boldsymbol{J}_n - \boldsymbol{W}_h^2)\boldsymbol{B})^i\boldsymbol{A}\boldsymbol{J}_n, \\
\bar{R}_{1,2} &= \boldsymbol{d}\sum_{i=0}^{s-1}(-1)^i\left\{(\tau^2\boldsymbol{W}_h^2\boldsymbol{A}\boldsymbol{B} - \tau^2\boldsymbol{A}\boldsymbol{J}_n\boldsymbol{B})^i - (\tau^2\boldsymbol{W}_h^2\boldsymbol{A}\boldsymbol{B})^i\right\}\boldsymbol{W}_h, \\
\bar{R}'_{1,1} &= -\tau\boldsymbol{d}'\boldsymbol{W}_h^2\sum_{i=0}^{s-1}(-1)^i\left\{(\tau^2\boldsymbol{W}_h^2\boldsymbol{A}'\boldsymbol{B}' - \tau^2\boldsymbol{A}'\boldsymbol{B}'\boldsymbol{J}_{n+1})^i - (\tau^2\boldsymbol{W}_h^2\boldsymbol{A}'\boldsymbol{B}')^i\right\}\boldsymbol{A}' \\
&\quad + \tau\boldsymbol{d}'\boldsymbol{J}_{n+1}\sum_{i=0}^{s-1}(\tau^2\boldsymbol{A}'\boldsymbol{B}'(\boldsymbol{J}_{n+1} - \boldsymbol{W}_h^2))^i\boldsymbol{A}', \\
\bar{R}'_{1,2} &= -\boldsymbol{d}'\boldsymbol{W}_h\sum_{i=0}^{s-1}(-1)^i\left\{(\tau^2\boldsymbol{W}_h^2\boldsymbol{A}'\boldsymbol{B}' - \tau^2\boldsymbol{A}'\boldsymbol{B}'\boldsymbol{J}_{n+1})^i - (\tau^2\boldsymbol{W}_h^2\boldsymbol{A}'\boldsymbol{B}')^i\right\} \\
&\quad + \boldsymbol{d}'\boldsymbol{J}_{n+1}\sum_{i=0}^{s-1}(\tau^2\boldsymbol{A}'\boldsymbol{B}'(\boldsymbol{J}_{n+1} - \boldsymbol{W}_h^2))^i\boldsymbol{W}_h^{-1}.
\end{aligned}
$$

By (31) and (H6), we can estimate $\bar{R}_{1,i},\ \bar{R}'_{1,i},\ i = 1,2$ as

$$
\bar{R}_{1,i} = O(\tau),\ \bar{R}'_{1,1} = O(\tau),\ \bar{R}'_{1,2} = O(1). \tag{39}
$$

Substituting (30) into (33) and (38), we get

$$\left|\left|\left(\bar{\xi}_n, \bar{\xi}_{n+1/2}\right)^T\right|\right|_{W_h} \leq C_1' \left(\tau^2 + \max_{i=0,1} ||\alpha_h(t_{n+i})||\right) \tag{40}$$

with a positive constant C_1'.
For $\theta \in S'$, there exit some positive constants $\gamma_4,\ \gamma_4'$ such that, $r_{1,2}(\theta)\mathbf{1}/\theta = d(I_s + \theta^2 AB)^{-1}\mathbf{1} > \gamma_4$ and $-r_{1,2}'(\theta)\mathbf{1}/\theta = d'(I_s + \theta^2 A'B')^{-1}\mathbf{1} > \gamma_4'$. By (H6), any eigenvalues of $[\boldsymbol{d}(\boldsymbol{I} + \tau^2 \boldsymbol{W}_h^2 \boldsymbol{AB})^{-1}\mathbf{1}']^{-1}$ and $[\boldsymbol{d}(\boldsymbol{I} + \tau^2 \boldsymbol{W}_h^2 \boldsymbol{AB})^{-1}\mathbf{1}']^{-1}$ are less than γ_4 and γ_4', respectively. Substituting (30) into (37), $W_h^{-1}\tau^{-1}W_h\psi_n$ and $W_h^{-1}\tau^{-1}\psi_{n+1/2}$ are represented as

$$\begin{aligned} W_h^{-1}\tau^{-1}W_h\psi_n &= r_{1,2}(\tau W_h)\boldsymbol{A}\boldsymbol{\alpha}_h(t_n) + \mathrm{O}(\tau^2), \\ W_h^{-1}\tau^{-1}\psi_{n+1/2} &= (r_{1,1}'(\tau W_h)\boldsymbol{B}' + \boldsymbol{d}')\boldsymbol{\alpha}_h(t_{n+1}) + \mathrm{O}(\tau^2). \end{aligned} \tag{41}$$

Substituting (41) into (34), there is a positive constant C_1'' such that

$$\left|\left|\left(\nu_n, \nu_{n+1/2}\right)^T\right|\right|_{W_h} \leq C_1'' \left(\tau^2 + \max_{i=0,1} ||\alpha_h(t_{n+i})||\right). \tag{42}$$

Since $\boldsymbol{u}_h^{(3)}(t_{n+1}) - \boldsymbol{u}_h^{(3)}(t_n) = O(\tau)$ and $\boldsymbol{v}_h^{(3)}(t_{n+3/2}) - \boldsymbol{v}_h^{(3)}(t_{n+1/2}) = O(\tau)$, we get

$$W^{-1}\tau^{-1}W_h(\psi_{n+1} - \psi_n) = \tau r_{1,2}(\tau W_h)\boldsymbol{A}\left\{\boldsymbol{\alpha}_h(t_{n+1}) - \boldsymbol{\alpha}_h(t_n)\right\} + \mathrm{O}(\tau^3),$$
$$W^{-1}\tau^{-1}(\psi_{n+3/2} - \psi_{n+1/2}) = \tau(r_{1,1}'(\tau W_h)\boldsymbol{B}' + \boldsymbol{d}')\left\{\boldsymbol{\alpha}_h(t_{n+2}) - \boldsymbol{\alpha}_h(t_{n+1})\right\} + \mathrm{O}(\tau^3).$$

Thus, by using (35), (40) and (42), there is a positive constant C_2 such that

$$\left|\left|\left(\hat{\xi}_n,\ \hat{\xi}_{n+1/2}\right)^T\right|\right|_{W_h} \leq C_2 \left(\tau^3 + \tau \max_{i=0,1} ||\alpha_h(t_{n+i})||\right). \tag{43}$$

Moreover, let ω_j be the eigenvalues of W_h. Then, by taking the orthogonal matrix P to be $P^{-1}(\tau W_h)P = \mathrm{diag}(\tau\omega_j)$, we have

$$R(\tau W_h) = \boldsymbol{P}R(\mathrm{diag}(\tau\omega_j))\boldsymbol{P}^{-1}, \text{ where } \boldsymbol{P} = I_2 \otimes P.$$

Here $R(\mathrm{diag}(\tau\omega_j))$ is the same formula as (10), replacing θ by $\mathrm{diag}(\tau\omega_j)$. Let $\lambda_\pm(\tau\omega_j) = \lambda_{\pm j}$ be the eigenvalues of $R(\mathrm{diag}(\tau\omega_j))$. $\lambda_{\pm j}$ are the solutions of (12), replacing θ by $\tau\omega_j$. By (H6), we have $0 \leq \tau\omega_j < \gamma_0$ and

$$|\lambda_{\pm j}| \leq 1\ ,\ j = 1, \cdots, m.$$

Then, by using Theorem 3.1, we obtain

$$||R(\tau W_h)^n|| = ||R(\text{diag}(\tau\omega_j))^n|| \le K \tag{44}$$

with K a constant independent of $n \in \mathbb{N}$, τ and h, $||\cdot||$ denotes the operator norm for $2m \times 2m$ matrices.
By (39), we obtain

$$||\bar{\boldsymbol{R}}_n|| \le K_1, \tag{45}$$

where K_1 is a constant independent of n, τ and h.
From (44) and (45), we obtain

$$\left|\left|\prod_{i=1}^{n} \boldsymbol{R}_i\right|\right| \le ||R(\tau W_h)^n||(1+\tau K_1)^n \le K e^{n\tau K_1} \le K_2. \tag{46}$$

Hence, from (36), (43) and (46), we obtain

$$\left|\left|\left(\hat{\varepsilon}_n, \hat{\varepsilon}_{n+1/2}\right)^T\right|\right|_{W_h} \le K_2 \left|\left|\left(\hat{\varepsilon}_0, \hat{\varepsilon}_{1/2}\right)^T\right|\right|_{W_h} + K_2 n C_2 \left(\tau^3 + \tau \max_{0\le t\le T} ||\alpha_h(t)||\right),$$

which implies that

$$\left|\left|\left(\hat{\varepsilon}_n, \hat{\varepsilon}_{n+1/2}\right)^T\right|\right|_{W_h} \le K_2 \left|\left|\left(\nu_0, \varepsilon_{1/2} + \nu_{1/2}\right)^T\right|\right|_{W_h} + K_2 T C_2 \left(\tau^2 + \max_{0\le t\le T} ||\alpha_h(t)||\right)$$

for $1 \le n \le N$. Using $\left|\left|\left(\nu_0, \varepsilon_{1/2} + \nu_{1/2}\right)^T\right|\right|_{W_h} = C_2'\tau^2$ for a constant $C_2' > 0$,

$$\left|\left|\left(\varepsilon_n, \varepsilon_{n+1/2}\right)^T\right|\right|_{W_h} \le \left|\left|\left(\hat{\varepsilon}_n, \hat{\varepsilon}_{n+1/2}\right)^T\right|\right|_{W_h} + \left|\left|\left(\nu_n, \nu_{n+1/2}\right)^T\right|\right|_{W_h}$$

and rewriting the constants, we finally obtain (28). □

5. Numerical experiments

We examine the convergence of the leapfrog scheme (13) and RKS4 (15), by using the following model problem of the form

$$\begin{aligned}
&\frac{\partial u}{\partial t} = v, \quad \frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(t,x,u), \quad 0 \le t \le T, \quad x \in \Omega, \\
&u(t,0) = \beta_0(t),\ u(t,1) = \beta_1(t), \quad 0 \le t \le T, \\
&u(0,x) = u_0(x),\ \frac{\partial u}{\partial t}(0,x) = v_0(x), \quad x \in \Omega.
\end{aligned} \tag{47}$$

Here $T = 1,\ \Omega = [0, 1],\ g(t, x, u) = -\sin u(t)$ and $\beta_0(t),\ \beta_1(t),\ u_0(x)$ and $v_0(x)$ are given by using the following exact solution ([13])

$$u(t,x) = 4\tan^{-1}\left\{\gamma \sinh\left(\frac{x}{\sqrt{1-\gamma^2}}\right) / \cosh\left(\frac{\gamma t}{\sqrt{1-\gamma^2}}\right)\right\}$$

with $\gamma = 0.5$. Let N be a positive integer, $h = 1/N$, and Ω_h be a uniform grid with nodes $x_j = jh,\ j = 0, 1, \cdots, N$. We discretize $\dfrac{\partial v}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + g(t, x, u)$ in space with the forth-order implicit scheme

$$\frac{1}{12}\left\{\frac{dv^{j-1}(t)}{dt} + 10\frac{dv^{j}(t)}{dt} + \frac{dv^{j+1}(t)}{dt}\right\} = \frac{1}{h^2}\left\{u^{j-1}(t) - 2u^{j}(t) + u^{j+1}(t)\right\}$$
$$-\frac{1}{12}\left\{\sin u^{j-1}(t) + 10\sin u^{j}(t) + \sin u^{j+1}(t)\right\}$$

with $u^j(t) \approx u(t, x_j)$, $v^j(t) \approx v(t, x_j)$ (see, [16]). Putting

$$u_h(t) = \left(u^0(t), \cdots, u^N(t)\right)^T,\ v_h(t) = \left(v^0(t), \cdots, v^N(t)\right)^T,$$

we obtain an MOL approximation

$$\frac{du_h(t)}{dt} = v_h(t), \quad \hat{H}\frac{dv_h(t)}{dt} = \hat{L_h}u_h(t) + \hat{\varphi}_h(t) + \hat{H}g_h(t, u_h(t)), \tag{48}$$

where

$$\hat{L_h} = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2 \end{pmatrix}, \quad \hat{H} = \frac{1}{12}\begin{pmatrix} 10 & 1 & 0 & \cdots & 0 \\ 1 & 10 & 1 & \cdots & 0 \\ 0 & 1 & 10 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 10 \end{pmatrix},$$

and $\hat{\varphi}_h(t) = (\beta_0(t),\ 0,\ \cdots,\ 0,\ \beta_1(t))^T$. The eigenvalues of $\hat{L}_h$ and $\hat{H}$ are

$$\frac{2}{h^2}\left(\cos\frac{(j+1)\pi}{N+2} - 1\right),\ \frac{1}{6}\left(5 + \cos\frac{(j+1)\pi}{N+2}\right), \quad j = 0, 1, \cdots, N, \tag{49}$$

respectively.
Multiplying $\hat{H}^{-1}$ to (48), we get (2) with $D = 1,\ L_h = \hat{H}^{-1}\hat{L}_h,\ \varphi_h(t) = \hat{H}^{-1}\hat{\varphi}_h(t)$. By (49) the eigenvalues of L_h is

$$\frac{12}{h^2}\left(1 - \frac{6}{5 + \cos((j+1)\pi/(N+2))}\right), \quad j = 0, 1, \cdots, N.$$

Since

$$\tau\rho(W_h) = \frac{2\sqrt{3}\tau}{h}\left(\frac{6}{5+\cos((N+1)\pi/(N+2))} - 1\right)^{\frac{1}{2}} < \frac{\sqrt{6}\tau}{h},$$

if we take the step size $\tau < \sqrt{2}h/\sqrt{3}$, (H6) holds for the leapfrog scheme. If we take the step size $\tau < 2h$, (H6) holds for RKS4. We take the various grid and step size of the form $h = 2\tau = 1/N$ so that both conditions are satisfied. We apply the leapfrog scheme and RKS4 to the MOL approximation (48), and integrate from $t = 0$ to $t = T$. We measure the errors of the schemes by using the discrete L_2-norm

$$\varepsilon_{u,L2} = \max_{0<n\leq 2NT} ||\varepsilon_n||, \ \varepsilon_{v,L2} = \max_{0<n\leq 2NT} ||\varepsilon_{n+1/2}||,$$

the discrete energy norm

$$\varepsilon_e = \max_{0<n\leq 2NT} ||(\varepsilon_n, \varepsilon_{n+1/2})||_{W_h}$$

and maximum norm

$$\varepsilon_{u,\max} = \max_{0<n\leq 2NT}\{||\varepsilon_n||_\infty\}, \ \varepsilon_{v,\max} = \max_{0<n\leq 2NT}\{||\varepsilon_{n+1/2}||_\infty\}$$

with $||\cdot||_\infty$ the maximum norm on $\mathbb{R}^m$.

Table 1: Numerical results for (47) using the leapfrog scheme

N	10	20	40	80	160	320	640
$-\log_2 \varepsilon_{u,L2}$	16.04	18.15	20.17	22.18	24.18	26.18	28.18
Increment		2.11	2.02	2.01	2.00	2.00	2.00
$-\log_2 \varepsilon_{v,L2}$	14.10	16.13	18.14	20.14	22.15	24.15	26.15
Increment		2.03	2.01	2.00	2.01	2.00	2.00
$-\log_2 \varepsilon_{u,\max}$	15.55	17.66	19.68	21.69	23.69	25.69	27.69
Increment		2.11	2.02	2.01	2.00	2.00	2.00
$-\log_2 \varepsilon_{v,\max}$	13.70	15.75	17.76	19.77	21.77	23.77	25.77
Increment		2.05	2.01	2.01	2.00	2.00	2.00
$-\log_2 \varepsilon_e$	13.31	15.40	17.42	19.42	21.43	23.43	25.43
Increment		2.09	2.02	2.00	2.01	2.00	2.00

Table 2: Numerical results for (47) using RKS4

N	10	20	40	80	160	320	640
$-\log_2 \varepsilon_{u,L2}$	19.17	23.16	27.15	31.15	35.15	39.15	43.14
Increment		3.99	3.99	4.00	4.00	4.00	3.99
$-\log_2 \varepsilon_{v,L2}$	18.28	22.27	26.23	29.67	32.24	34.74	37.14
Increment		3.99	3.96	3.44	2.57	2.50	2.40
$-\log_2 \varepsilon_{u,\max}$	18.73	22.71	26.70	30.70	34.70	38.70	42.62
Increment		3.98	3.99	4.00	4.00	4.00	3.92
$-\log_2 \varepsilon_{v,\max}$	17.51	21.51	24.62	26.60	28.59	30.59	32.57
Increment		4.00	3.11	1.98	1.99	2.00	1.98
$-\log_2 \varepsilon_{e}$	16.98	20.97	24.90	28.60	31.75	34.55	36.73
Increment		3.99	3.93	3.70	3.15	2.80	2.18

Table 1 and Table 2 show that the observed order of the leapfrog scheme and RKS4 is more than or equal 2. We observe that the order for u of RKS4 is higher than expected results from Theorem 4.1.

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